In this paper, we propose a Markov chain modeling of complicated phenomena observed from coupled chaotic oscillators. Once we obtain the transition probability matrix from computer simulation results, various statistical quantities can be easily calculated from the model. It is shown that various statistical quantities are easily calculated by using the Markov chain model. Various features derived from the Markov chain models of chaotic wandering of synchronization states and switching of clustering states are compared with those obtained from computer simulations of original circuit equations.

Keywords: Coupled oscillators; Markov chain; chaos; synchronization; clustering.

1. Introduction

Spatio-temporal phenomena observed from coupled chaotic systems attract many researchers’ attentions. For discrete-time mathematical models, there have been numerous excellent results. Kaneko’s coupled map lattice is the most interesting and well-studied system.\(^1\) He discovered various nonlinear spatio-temporal chaotic phenomena. Also Aihara’s chaos neural network is one of the most important chaotic network from an engineering point of view.\(^2\) His study indicated new possibility of engineering applications of chaotic networks, namely dynamical search of patterns
embedded in neural networks utilizing chaotic wandering. Furthermore, the application of chaos neural network to optimization problems is widely studied (Ref. 3 and references therein). On the other hand, for continuous-time systems, several results on arrays of Chua’s circuits have been reported (e.g., some papers in Ref. 4). However, the main subject of many studies has been wave propagation phenomenon observed for a given set of initial patterns and there are few studies on spatial patterns observed after vanishing effects of the initial patterns. Namely, the pattern switching caused by chaotic wandering as observed in Aihara’s discrete-time chaos neural network or switching of clustering states as observed in Kaneko’s coupled map lattice have not yet been studied well in continuous-time coupled oscillators networks. Therefore, in order to fill the gap between studies of discrete-time mathematical abstract and studies of continuous-time real physical systems, it is important to investigate simple continuous-time coupled chaotic oscillators generating chaotic wandering, clustering, pattern switching, and so on.

We have proposed continuous-time coupled chaotic oscillators and have investigated generating spatial patterns and chaotic wandering of spatial patterns. We also reported that similar coupled oscillators could generate chaotic wandering over several phase states characterized by four-phase synchronization and have shown that dependent variables corresponding to the angles of the solutions in subcircuit were useful to grasp the complicated phenomena. The important feature of our coupled chaotic oscillators is their coupling structure. Namely, several adjacent chaotic oscillators are coupled by one element. Because such a coupling exhibited quasi-synchronization with phase difference, various spatial patterns could be generated. It would be followed by the generation of several complicated spatio-temporal chaotic phenomena observed in discrete-time mathematical models. However, because it is extremely difficult to treat higher-dimensional nonlinear phenomena in continuous-time systems theoretically, we have to develop several tools to reveal the essence of the complicated phenomena.

In this paper, we propose a Markov chain modeling of complicated phenomena observed in coupled chaotic oscillators. Firstly, chaotic wandering phenomenon observed from four simple chaotic oscillators coupled by one resistor is modeled by first-order and second-order Markov chains with 6 states plus 3 intermediate states characterized by phase difference of synchronization states. It is shown that various statistical quantities are easily calculated by using the Markov chain model. Secondly, switching of clustering states observed from six simple chaotic oscillators coupled by one inductor is modeled by first-order Markov chain with 9 states corresponding to different clustering types characterized by the combinations of in-phase and anti-phase synchronizations. It is shown that the Markov chain modeling is effective to analyze these complicated phenomena observed from coupled chaotic oscillators. The modeling method using the Markov chain may contribute to the analysis of large scale oscillatory networks which are one of recent hot topics in the research fields of nonlinear circuits, neural networks and physics.
2. Resistively Coupled Chaotic Oscillators

Figure 1 shows the circuit diagram. In the circuit, four identical chaotic oscillators are coupled by one resistor $R$. Each chaotic oscillator is a three-dimensional autonomous circuit and consists of three memory elements, one linear negative resistor and one diode. We can regard the diodes as purely resistive elements, because their operation frequency is not too high. The coupling structure is symmetric in the sense that the exchange of any two subcircuits does not cause any change of the system structure. Also the coupling is complete in the sense that a signal of one oscillator can reach the others without passing through the rest.

The $i - v$ characteristics of the diodes are approximated by two-segment piecewise-linear functions as

$$v_d(i_k) = \frac{1}{2} (r_d i_k + E - |r_d i_k - E|).$$

By changing the variables and parameters,

$$I_k = \sqrt{\frac{C}{L_1}} E x_k, \quad i_k = \sqrt{\frac{C}{L_1}} E y_k, \quad v_k = E z_k, \quad t = \sqrt{L_1 C} \tau,$$

$$\alpha = \frac{L_1}{L_2}, \quad \beta = r \sqrt{\frac{C}{L_1}}, \quad \gamma = R \sqrt{\frac{C}{L_1}}, \quad \delta = r_d \sqrt{\frac{C}{L_1}},$$

$(k = 1, 2, 3, 4)$,
the normalized circuit equations are given as

\[
\frac{dx_k}{d\tau} = \beta (x_k + y_k) - z_k - \gamma \sum_{j=1}^{4} x_j ,
\]

\[
\frac{dy_k}{d\tau} = \alpha \{ \beta (x_k + y_k) - z_k - f(y_k) \} ,
\]

\[
\frac{dz_k}{d\tau} = x_k + y_k ,
\]

\[(k = 1, 2, 3, 4) ,
\]

where

\[
f(y_k) = \frac{1}{2} (\delta y_k + 1 - |\delta y_k - 1| ) .
\]

3. Four-Phase Synchronization and Chaotic Wandering

In this section, chaotic wandering of phase states observed from the circuit, which has been reported in our previous study, is briefly introduced.

Figure 2 shows an example of the four-phase synchronizations of chaotic oscillation observed from the circuit in Fig. 1. Though only circuit experimental results are shown, similar results can be obtained by computer calculations. In the figures

(a) 
(b) 
(c) 
(d) 
(e) 

Fig. 2. Four-phase synchronization of chaos. \(L_1 = 100.7 \text{ mH}, L_2 = 10.31 \text{ mH}, C = 34.9 \text{ nF}, r = 334 \Omega\) and \(R = 198 \Omega\). (a) \(I_1-I_2\), (b) \(I_1-I_3\), (c) \(I_1-I_4\), (d) \(I_1-v_1\), (e) Time waveforms.
the phase differences of $I_2$, $I_3$ and $I_4$ with respect to $I_1$ are almost 90°, 180° and 270°, respectively. Because of the symmetry of the coupling structure, six different combinations of phase states coexist and they can be observed by giving different initial conditions to the circuit.

Further, we could observe chaotic wandering of the six phase states of the four-phase synchronization by tuning the coupling parameter value. For such parameter values, all of the six phase states become unstable and the solution starts wandering over the six phase states. Although the wandering speed depends significantly on the parameter value, we could observe in the circuit experiments that one phase state switches to another within one second or after 10 seconds. The wandering looks truly chaotic, i.e., we cannot predict when the next switching will occur or which phase state will appear next.

In order to observe the chaotic wandering clearly, we defined the Poincaré section as $z_1 = 0$ and $x_1 < 0$. Further we defined the following dependent variables from the discrete data of $x_k(n)$ and $z_k(n)$ on the Poincaré map.

$$\varphi_k(n) = \begin{cases} 
\pi - \tan^{-1}\frac{z_{k+1}(n)}{x_{k+1}(n)} \cdots x_{k+1}(n) \geq 0 \\
-\tan^{-1}\frac{z_{k+1}(n)}{x_{k+1}(n)} \cdots x_{k+1}(n) < 0 \text{ and } z_{k+1}(n) \geq 0 \\
2\pi - \tan^{-1}\frac{z_{k+1}(n)}{x_{k+1}(n)} \cdots x_{k+1}(n) < 0 \text{ and } z_{k+1}(n) < 0 
\end{cases}$$

(k = 1, 2, 3).

Because the attractor observed from each subcircuit is strongly constrained onto the plane $y_k = 0$ when the diode is off, these variables can correspond to the phase differences between the subcircuit 1 and the others. (Note that the argument of the point $(x_1(n), z_1(n))$ is always $\pi$.) Figure 3 shows an example of time evolutions of $\varphi_k(n)$ when the chaotic wandering occurs. We can see that several switchings of the phase difference appear in an irregular manner.

![Fig. 3. Chaotic wandering of phase states. $\alpha = 7.0$, $\beta = 0.13$, $\gamma = 0.46$ and $\delta = 50.0.$](image-url)
4. Markov Chain Modeling of Chaotic Wandering

In our previous study, some statistical features of the chaotic wandering has been calculated by using the depending variables in Eq. (5). However, in order to exploit the phenomena for future engineering applications, we have to make a simple model to extract important features of the phenomena and to calculate necessary information with required accuracy and speed. Therefore, in this paper, we propose a method using the Markov chain to model the chaotic wandering observed in the coupled chaotic oscillators.

4.1. Definition of phase states

Let us define the six phase states of the four-phase synchronization concretely using the variables in Eq. (5) as

\[ S_1 : \varphi_1 < \varphi_2 < \varphi_3, \]
\[ S_2 : \varphi_1 < \varphi_3 < \varphi_2, \]
\[ S_3 : \varphi_2 < \varphi_1 < \varphi_3, \]
\[ S_4 : \varphi_2 < \varphi_3 < \varphi_1, \]
\[ S_5 : \varphi_3 < \varphi_1 < \varphi_2, \]
\[ S_6 : \varphi_3 < \varphi_2 < \varphi_1. \]  

By careful investigation of the behavior of the solutions around switchings between two of the above six phase states, we have found that the solution often stays in the intermediate phase states during a certain period. In the intermediate phase states, 2 of the four oscillators are almost synchronized at in-phase and the rest are also synchronized at in-phase with $\pi$-phase difference against the former pair (like in- and opposite-phases quasi-synchronization in Ref. 5). These phase states can be characterized by

\[ S_{I1} : \min\{2\pi - \varphi_1, \varphi_1\} < \theta_I \cap |\varphi_2 - \varphi_3| < \theta_I, \]
\[ S_{I2} : \min\{2\pi - \varphi_2, \varphi_2\} < \theta_I \cap |\varphi_1 - \varphi_3| < \theta_I, \]
\[ S_{I3} : \min\{2\pi - \varphi_3, \varphi_3\} < \theta_I \cap |\varphi_1 - \varphi_2| < \theta_I, \]  

where $\theta_I$ is a parameter deciding the size of the region of the intermediate phase states. Note that the conditions of the decisions corresponding to the original six phase states in Eq. (6) are modified so that their regions do not overlap with those of Eq. (7).

4.2. State-transition diagram

Next, let us consider a state-transition diagram representing the transitions between the above-mentioned phase states $S_1 - S_6$ and $S_{I1} - S_{I3}$. By virtue of the symmetry of the coupling structure of the original circuit, it is enough to consider only $S_1$ and $S_{I1}$.
Figure 4 shows a part of the state-transition diagram focusing on the phase state $S_1$ where

$$ P_1 = 1 - \sum_{i=2}^{7} P_i. $$

(8)

Note that the paths from $S_1$ to $S_6$ and to $S_{I2}$ do not exist, because these transitions need double switching beyond one of the other phase states which was very rare in our computer simulations.

Figure 5 shows a part of the state-transition diagram focusing on the intermediate phase state $S_{I1}$ where

$$ P_8 = 1 - 2(P_9 + P_{10}). $$

(9)

Because of the symmetry, the transition probabilities from $S_{I1}$ to $S_1$ and to $S_2$ can be assumed to be the same. Also to $S_1$ and to $S_6$ are assumed to be the same. The reason of the missing paths to $S_3$, $S_5$, $S_{I2}$ and $S_{I3}$ is the very rare double switching again.

From the whole state-transition diagram, we can derive the transition probability matrix $\mathbf{P}$ as

$$
\mathbf{P} = 
\begin{bmatrix}
P_1 & P_2 & P_3 & P_5 & P_4 & 0 & P_9 & 0 & P_{10} \\
P_2 & P_4 & P_4 & 0 & P_3 & P_5 & P_9 & P_{10} & 0 \\
P_3 & P_5 & P_1 & P_2 & 0 & P_4 & 0 & P_9 & P_{10} \\
P_4 & 0 & P_2 & P_1 & P_5 & P_3 & P_{10} & P_9 & 0 \\
P_5 & P_3 & 0 & P_4 & P_1 & P_2 & 0 & P_{10} & P_9 \\
0 & P_4 & P_5 & P_3 & P_2 & P_1 & P_{10} & 0 & P_9 \\
P_6 & P_6 & 0 & P_7 & 0 & P_7 & P_8 & 0 & 0 \\
0 & P_7 & P_6 & P_6 & P_7 & 0 & 0 & P_8 & 0 \\
P_7 & 0 & P_7 & 0 & P_6 & P_6 & 0 & 0 & P_8
\end{bmatrix}.
$$

(10)
Since $P_1$ and $P_8$ can be calculated by Eqs. (8) and (9), the Markov chain model of the chaotic wandering can be described by 8 transition probabilities.

4.3. Basic quantities

The stationary probability distribution describing probability of the solution being in each phase state

$$Q = [Q_{S_1}, Q_{S_2}, \ldots, Q_{S_6}]^T$$

(11)

can be calculated from the following equation

$$Q = PQ$$

(12)

with

$$\sum_{i=1}^{6} Q_{S_i} + \sum_{j=1}^{3} Q_{S_{ij}} = 1.$$  

(13)

It is also possible to estimate the expected sojourn time in each phase state by using the transition probabilities. For example, the probability density function of the sojourn time in $S_1$ is given by

$$P_{ST}(S_1, n) = P_1^{n-1}(1 - P_1).$$

(14)
From Eq. (14) the expected sojourn time in $S_1$ is calculated as 

$$E_{ST}(S_1) = \sum_{n=1}^{\infty} \{n \times P_{ST}(S_1, n)\}$$

$$= (1 - P_1) \sum_{n=1}^{\infty} n P_1^{n-1}$$

$$= (1 - P_1)(1 + 2P_1 + 3P_1^2 + \cdots )$$

$$= (1 - P_1) \frac{d}{dP_1} (1 + P_1 + P_1^2 + \cdots )$$

$$= (1 - P_1) \frac{d}{dP_1} \left( \frac{1}{1 - P_1} \right)$$

$$= \frac{1}{1 - P_1}.$$  \hfill (15)

### 4.4. Second-order Markov chain

In order to obtain more accurate statistical information of the phenomena, we also model the phenomenon by second-order Markov chain.

In the second-order Markov chain, each state of the chain $S_2^{(2)}$ corresponds to two successive phase states of chaotic wandering, for example $\{S_1^{(2)} : (S_1 \to S_1)\}$, $\{S_2^{(2)} : (S_1 \to S_2)\}$, $\{S_{T_2}^{(2)} : (S_1 \to S_{T_2})\}$, and so on. Hence, if there is no direct transition path between two phase states, the state of the chain corresponding to the transition does not exist, for example, state of the chain corresponding to $(S_1 \to S_6)$ or $(S_{T_1} \to S_3)$ does not exist. This means that the number of the states of the chain is determined by the number of the transition paths of the state-transition diagram in Figs. 4 and 5. From those figures, the number of the transition paths from the phase states $S_i$ ($i = 1 \sim 6$) is 7 and the number from the intermediate phase states $S_{T_j}$ ($j = 1 \sim 3$) is 5. Therefore, the total number of the states of the second-order Markov chain can be calculated as $6 \times 7 + 3 \times 5 = 57$.

This immediately gives the size of the transition probability matrix of the second-order Markov chain as $57 \times 57$. However, many of the elements of the matrix becomes zero. Because those elements correspond to non-existing transition, for example a state $(S_1 \to S_2)$ to a state $(S_1 \to S_3)$. In other words, a state $(S_1 \to S_2)$ should move to a state $(S_2 \to S_{any})$. From this restriction, the number of nonzero value is only 369 out of $57 \times 57 = 3249$. Further, by virtue of the symmetry of the coupling structure of the original circuit, the second-order Markov chain can be described by 52 different transition probabilities.

### 5. Results and Discussions

The transition probabilities of the Markov chain model of the chaotic wandering significantly depend on the parameters of the original circuit. In this study, those
values are obtained by counting all of the transitions during 10,000,000 iterations of the Poincaré map. Note that once we obtain the transition probability matrix $P$, the simulations using the Markov chain model is very easy and various characteristics of the chaotic wandering can be calculated.

Table 1 shows the stationary probabilities and the expected sojourn times. For computer simulations, we have to integrate the original differential Eq. (3) numerically by using the Runge–Kutta method, hence it takes a very long time even for these basic quantities to obtain convergent average values. On the other hand, for the first-order and second-order Markov chains, even simulations of the Markov chain are not necessary. We obtain the quantities of the first-order Markov chain by using Eqs. (12)–(15). The second-order Markov chain also gives these quantities after similar calculations using the transition probability matrix.

We can see that the results obtained from both of the first-order and second-order Markov chains agree very well with computer simulated results. However, we should note that the values of $E_{ST}$ are only averaged values. Namely, closer examinations are necessary to know more detailed statistical behaviors.

Figure 6 shows the probability density functions of the sojourn time. From the figures, we can see that the first-order Markov chain could not explain the chaotic wandering phenomenon in the circuit correctly. On the other hand, the second-order Markov chain displays better agreement especially for the intermediate states $S_{11} - S_{13}$ (Fig. 6(2)). So, we can say that the both the first-order and the second-order Markov chains could give a good agreement in terms of the averaged value. However, the first-order Markov chain is not enough to model the detailed statistical features. Further, we have to mention that the error for $S_1 - S_6$ around $n = 2, 3$ increases as $\theta_j$ increases (Fig. 6(1c)). This error is considered to be caused by unneglectable higher-order
Fig. 6. Probability density functions of sojourn time. $\alpha = 7.0$, $\beta = 0.13$, $\gamma = 0.50$ and $\delta = 50.0$.

(a) $\theta_I = \pi/12$, (b) $\theta_I = \pi/6$, (c) $\theta_I = \pi/4$, (1) $S_1 - S_6$, (2) $S_{11} - S_{13}$. 

Markov Chain Modeling and Analysis of Complicated Phenomena
Markov property. Investigating the relationship between the error and the order of the Markov chains is our important future problem.

6. Markov Chain Modeling of Clustering Phenomena

In this section, we show that the Markov chain modeling is also effective for the analysis of clustering phenomena observed from different type of coupled chaotic oscillators.

6.1. Inductively coupled chaotic oscillators

Figure 7 shows inductively coupled chaotic oscillators. In the circuit, \( N \) identical chaotic oscillators are coupled symmetrically by an inductor \( M \). Each chaotic oscillator is a symmetric version of the chaotic oscillator used in Fig. 1. First, we approximate the \( i - v \) characteristics of the nonlinear resistors consisting of diodes by the following function;

\[
v_{d}(i_{k}) = \sqrt[2]{r_{d}i_{k}}. \tag{16}
\]

By changing the variables and parameters,

\[
a = \sqrt{\frac{r_{d}C}{L_{1}}}, \quad I_{k} = a \sqrt{\frac{C}{L_{1}}} x_{k}, \quad i_{k} = a \sqrt{\frac{C}{L_{1}}} y_{k}, \quad v_{k} = az_{k},
\]

\[
t = \sqrt{L_{1}C}, \quad \alpha = \frac{L_{1}}{L_{2}}, \quad \beta = r \sqrt{\frac{C}{L_{1}}}, \quad \gamma = \frac{M}{L_{1} + M}, \quad (k = 1, 2, \ldots, N), \tag{17}
\]

![Inductively coupled chaotic oscillators](image)
the normalized circuit equations are given as

\[
\frac{dx_k}{d\tau} = \beta(x_k + y_k) - z_k - \frac{\gamma}{1 + (N-1)\gamma} \sum_{j=1}^{N} \{\beta(x_j + y_j) - z_j\},
\]

\[
\frac{dy_k}{d\tau} = \alpha\{\beta(x_k + y_k) - z_k - f(y_k)\},
\]

\[
\frac{dz_k}{d\tau} = x_k + y_k,
\]

\[(k = 1, 2, \ldots, N),\]

where

\[
f(y_k) = \sqrt{y_k}.
\]

6.2. Clustering phenomena

Figure 8 shows an example of computer calculated results for the case of \(N = 6\). We can observe that the generation of clustering phenomena whose states are characterized by combinations of in-phase synchronization and anti-phase synchronization. Also, we can confirm the switching of the clustering states.

In order to model this complicated phenomena by Markov chain, we apply the dependent variables (5) again. By using the values of \(\varphi_k(n)\), we can classify the
clustering states into the following 11 different types.

\[
\begin{align*}
S_1 : & \ 1 - 1 - 1 - 1 - 1 - 1, \\
S_2 : & \ 2 - 1 - 1 - 1 - 1, \\
S_3 : & \ 2 - 2 - 1 - 1, \\
S_4 : & \ 2 - 2 - 2, \\
S_5 : & \ 3 - 1 - 1 - 1, \\
S_6 : & \ 3 - 2 - 1, \\
S_7 : & \ 3 - 3, \\
S_8 : & \ 4 - 1 - 1, \\
S_9 : & \ 4 - 2, \\
S_{10} : & \ 5 - 1, \\
S_{11} : & \ 6.
\end{align*}
\]

In this notation, the number means the oscillators synchronized at in-phase. For example, \(S_{11}\) corresponds to the state that all 6 oscillators are synchronized at in-phase (complete synchronization), \(S_7\) corresponds to the state that 6 oscillators are divided into two groups of 3 synchronized oscillators, and \(S_1\) corresponds to the state that any pairs of the 6 oscillators are not synchronized at in-phase.

6.3. Markov chain modeling

The state-transition diagram representing the transitions among the 9 clustering types \(S_1 - S_9\) is shown in Fig. 9. We omit the two clustering types \(S_{10}\) and \(S_{11}\) from the diagram, because we could not observe these two clustering types during computer simulations for the parameter values at which the switching of the clustering states occurs.

From the whole state-transition diagram, we can derive the transition probability matrix \(P\) as

\[
P = \begin{bmatrix}
P(S_1|S_1) & \cdots & P(S_1|S_9) \\
\vdots & \ddots & \vdots \\
P(S_9|S_1) & \cdots & P(S_9|S_9)
\end{bmatrix}.
\]

\[\text{Fig. 9. State-transition diagram (only transitions from } S_1).\]
The conditional probability $P(S_k|S_l)$ denotes the transition probability from the cluster type $S_l$ to the cluster type $S_k$. These values are obtained by counting all of the transitions during computer simulations.

Figure 10 shows the reproduction of the switching of clustering states by using the Markov chain model. Figure 10(a) is an example of switchings of clustering types observed from computer simulations of the original differential equations (18). While Figure 10(b) is an example of the switchings reproduced from the Markov chain model. We can see that the Markov chain model can successfully imitate the complicated switching phenomena.

Further, similar to the results in Sec. 5, basic quantities of the Markov chain model can be easily calculated by using the values of the transition probability matrix $P$. Table 2 shows the stationary probabilities and the expected sojourn times obtained from computer simulations of (18) and the Markov chain model. As we can see from Table 2, the results obtained from the Markov chain model agree with computer simulated results.
Table 2. Stationary probabilities $Q$ and expected sojourn times $E_{ST}$. $\alpha = 20.0$, $\beta = 0.265$ and $\gamma = 0.3$.

<table>
<thead>
<tr>
<th>Cluster type</th>
<th>Simulation $Q_S$</th>
<th>Markov $Q_S$</th>
<th>Simulation $E_{ST}(S_k)$</th>
<th>Markov $E_{ST}(S_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$: 1-1-1-1-1-1</td>
<td>0.00204</td>
<td>0.00204</td>
<td>1.36000</td>
<td>1.36000</td>
</tr>
<tr>
<td>$S_2$: 2-1-1-1-1</td>
<td>0.03006</td>
<td>0.03006</td>
<td>1.62662</td>
<td>1.62662</td>
</tr>
<tr>
<td>$S_3$: 2-2-1-1</td>
<td>0.10004</td>
<td>0.10004</td>
<td>1.63839</td>
<td>1.63839</td>
</tr>
<tr>
<td>$S_4$: 2-2-2</td>
<td>0.00597</td>
<td>0.00597</td>
<td>1.60484</td>
<td>1.60484</td>
</tr>
<tr>
<td>$S_5$: 3-1-1-1</td>
<td>0.08581</td>
<td>0.08581</td>
<td>1.65816</td>
<td>1.65816</td>
</tr>
<tr>
<td>$S_6$: 3-2-1</td>
<td>0.51865</td>
<td>0.51865</td>
<td>3.14677</td>
<td>3.14677</td>
</tr>
<tr>
<td>$S_7$: 3-3</td>
<td>0.25662</td>
<td>0.25662</td>
<td>2.50361</td>
<td>2.50361</td>
</tr>
<tr>
<td>$S_8$: 4-1-1</td>
<td>0.00023</td>
<td>0.00023</td>
<td>1.04545</td>
<td>1.04545</td>
</tr>
<tr>
<td>$S_9$: 4-2</td>
<td>0.00058</td>
<td>0.00058</td>
<td>1.07407</td>
<td>1.07407</td>
</tr>
</tbody>
</table>

Fig. 11. Probability density functions of sojourn time. $\alpha = 20.0$, $\beta = 0.265$ and $\gamma = 0.3$. 
Figure 11 shows the probability density functions of sojourn time for each cluster type. From the figures, we can say that the Markov chain model can explain the clustering phenomena efficiently. However, for the case of the cluster types $S_1$, $S_3$ and $S_5$, the error between computer simulations and the Markov chain model is large around $n = 1, 2$. Furthermore, the error is also generated around $n = 2, 4$ for $S_6$. The determination of the reason of these errors is also our future research.

7. Conclusions
In this paper, we have proposed a Markov chain modeling of complicated phenomena observed from coupled chaotic oscillators. Once we obtain the transition probability matrix from computer simulation results, we could easily calculate various statistical quantities of the phenomena. Chaotic wandering of synchronization states were modeled by the first-order and the second-order Markov chains and the calculated statistical quantities were compared with those obtained from computer simulations of the original circuit equations. Switching of clustering states was also modeled by the first-order Markov chain and the modeling method using Markov chain was confirmed to be also effective for the analysis of clustering phenomena.

We consider that the proposed approach can be applied for any kinds of complicated phenomena based on chaotic wandering, if all possible states of the system
can be represented by discrete states and their transition probabilities can be estimated by computer simulations or real experiments. Hence, we would like to extend this approach to the analysis and control of larger size of chaotic circuit networks in future.

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