Research on Synchronization Phenomena in Complex Networks by Using Oscillators Considering Euclidean Distance

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***Abstract*—Complex networks have attracted a great deal of attention in recent years. Because they have many properties that are relevant to real-world networks. This has been studied a lot in terms of network topology and interactions between nodes. However, most of these studies have analyzed networks by coupling strength is kept constant between nodes for all connections. In this study, we focus on the Euclidean distance between nodes to weight the connections. Coupling strength is determined by it. Two networks, a random network and a scale-free network, were constructed by using 100 oscillators. These networks were analyzed in terms of synchronization phenomena. As a result, it was confirmed that the synchronization phenomena in the two networks is significantly different when the coupling strength is determined by Euclidean distance.**

I. Introduction

Complex networks have the properties of real-world networks: scale-free, small-world, and clustered. They can be represented in graph theory and modeled by nodes and edges. Examples of these relationships are human relationships, transportation networks, neural networks and the Internet. Complex networks have received much attention in various fields such as sociology, biology and engineering. Furthermore, in the field of engineering, complex networks using circuits have been studied, and interesting phenomena such as synchronization between circuits have been observed [1]. Synchronization is one of the most familiar phenomena that exist in nature. It is a phenomena in which nonlinear systems that have been moving separately interact with each other and become aligned. Synchronization phenomena in complex networks have been studied in the past. These studies have focused on the topological structure of the network and investigated various synchronization phenomena caused by differences in the topological structure [2]. From the results of these studies, it was confirmed that the topological structure of the network influences the synchronization. However, most of these studies have examined synchronization phenomena in networks where the coupling strength is kept constant for all the connections [3]. Therefore we investigate how the synchronization phenomena differ when the coupling strength is considered as the Euclidean distance of each coupled part depending on the node arrangement.

In this study, two network models with different properties are constructed using 100 oscillators. For each network, the node configuration is placed on the two-dimensional space and the Euclidean distance is derived to set the coupling strength. The synchronization between the nodes of the coupled part is investigated by the computer simulations.

II. Network models

In this study, we use two network models, Barabási Albert model (BA model) [4] and Erdős Rényi model (ER model) [5]. The number of nodes is 100 and the average degree of each network model is set to be close to 4.0. Since the Euclidean distance between nodes determines the coupling strength, these two network models are constructed with fixed node configurations.

First, Fig. 1 shows the BA model, a model for scale-free networks used in this study. This network has properties called network growth and preferential selection, which is the property that hubs are created as new nodes are easily connected to nodes with higher degree. Furthermore, since the degree distribution follows a power law, a small number of nodes are connected to many other nodes and have a large degree, while many other nodes are connected to only a small number of nodes.

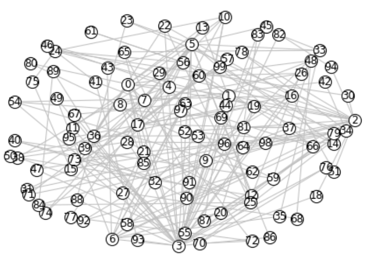


Fig. 1. BA model.

Next, Fig. 2 shows the ER model, a model for random networks used in this study. In this network, all pair nodes are connected with the same probability. Furthermore, since the degree distribution follows a binomial distribution, the majority of nodes have a degree close to the average degree.

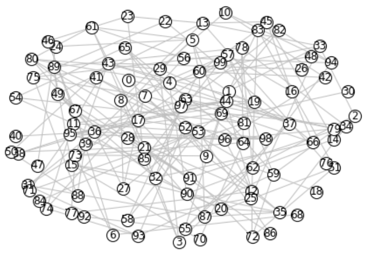


Fig. 2. ER model.

III. System model

Figure 3 shows a van der Pol oscillator. This oscillator is a simple circuit, consisting of only a capacitor, an inductor and a nonlinear element. This circuit is considered as a single node. By connecting these circuits with resistors, edges are created to form each network.

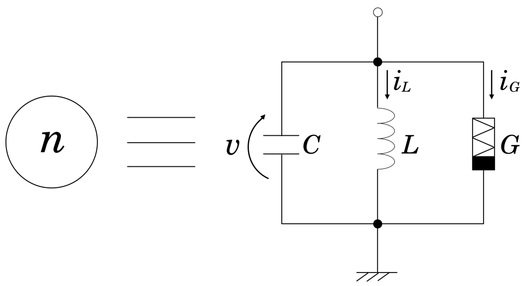


Fig. 3. van der Pol oscillator.

The circuit equation is described as follows:

$$\begin{cases} C\dfrac{dv_n}{dt} = -i_{L_n} - i_{g_n} - \displaystyle\sum_{n,k=1}^{100} \dfrac{1}{R_{nk}}(v_n - v_k) \\ L\dfrac{di_n}{dt} = v_n \\ \qquad\qquad\qquad (n, k = 1, 2, \cdots, 100). \end{cases} \tag{1}$$

Here, the parameter i_g is the equation of the nonlinear element, and described as follows:

$$i_g = -g_1 v + g_3 v^3. \tag{2}$$

By using the normalization parameter and the variables as follows:

$$v = \sqrt{\frac{g_1}{g_3}}x,\ i = \sqrt{\frac{g_1 C}{g_3 L}}y,\ t = \sqrt{LC}\tau$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}},\ \gamma = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

The normalized circuit equation described as follows:

$$\begin{cases} \dfrac{dx_n}{d\tau} = \alpha\left\{\varepsilon x_n(1 - x_n^2) - y_n - \displaystyle\sum_{n,k=1}^{100} E_{nk}\gamma_{nk}(x_n - x_k)\right\} \\ \dfrac{dy_n}{d\tau} = x_n \\ \qquad\qquad\qquad (n, k = 1, 2, \cdots, 100). \end{cases} \tag{3}$$

Here, The parameter of the van der Pol oscillator is set to $\varepsilon = 0.1$. Moreover α represents the small error of the capacitor. The results of the synchronization rate in the two network models are shown in the range of [0.95:1.05] in increments of 0.001. Further, E_{nk} represents the adjacency matrix of the network. This is a matrix that indicates whether node n and node k are connected or not. $E_{nk} = 1$ if node n and node k are connected, and $E_{nk} = 0$ if they are not connected.

The coupling strength γ_{nk} is determined by using the parameter q as follows:

$$\gamma_{nk} = \frac{1}{R_{nk}}\sqrt{\frac{L}{C}} = \frac{q}{{d_{nk}}^2}. \tag{4}$$

The parameter q is the weight of parameter that determines the coupling strengths [6]. In this case, we set parameter $q = 0.01$.

IV. Results

In order to investigate the synchronization phenomena in the coupled parts, we define the synchronization between circuits as follows:

$$|x_n - x_k| < 0.01. \tag{5}$$

Where n and k are the number of circuits. Figure 4 shows the differential voltage waveform observed in this study. The two lines in Fig. 4 represent the threshold value shown in Eq. (5). When the voltage difference meets this threshold, the circuits are considered to be synchronized. This is the voltage difference in the range of $\tau = 10000$ to $\tau = 20000$ where the system settles into a steady state, and the synchronization rate is calculated under these conditions.

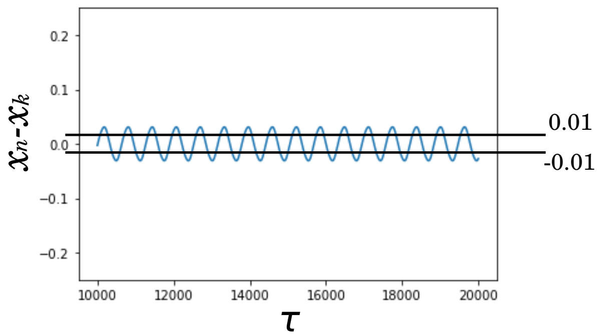


Fig. 4. The differential voltage waveform.

First, Fig. 5 shows the relationship between the synchronization rate and the number of pair nodes when the common coupling strength is kept constant for the BA model and ER model. Here, the coupling strength is set to $\gamma \simeq 0.3276$. Table I shows the average synchronization rate for each network under these conditions.

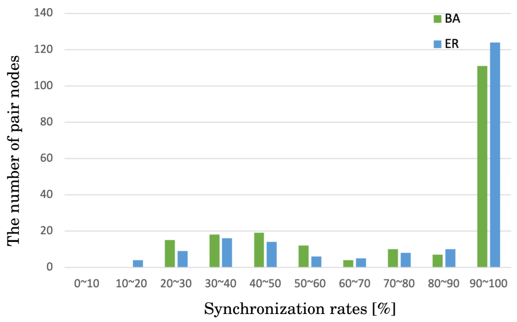


Fig. 5. Relationship between the synchronization rate and the number of pair nodes in the BA model and ER model with the same coupling strength.

TABLE I
AVERAGE SYNCHRONIZATION RATES OF TWO NETWORK MODELSWITH THE SAME COUPLING STRENGTH.

Network	BA model	ER model
Average synchronization rates[%]	77.674	81.074

Next, Fig. 6 shows the relationship between the synchronization rate and the number of pair nodes when the coupling strength is determined by Euclidean distance between nodes for the BA model and ER model using Eq. (4). Table Ⅱ also shows the average synchronization rate of each network under these conditions.

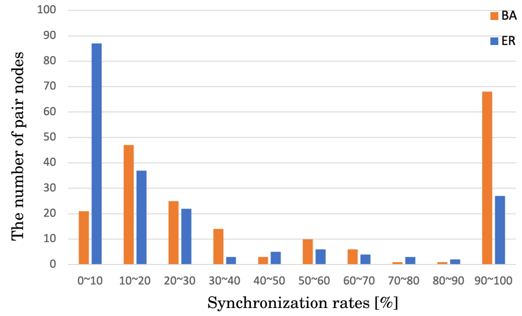


Fig. 6. Relationship between the synchronization rate and the number of pair nodes in the BA model and ER model determined by Euclidean distance.

TABLE II
AVERAGE SYNCHRONIZATION RATES OF TWO NETWORK MODELS.

Network	BA model	ER model
Average synchronization rates[%]	50.811	28.346

Finally, Fig. 7 and Fig. 8 show the results to compare the case of coupling strength kept constant at the average value obtained from Eq. (4) with the case of coupling strength determined by Euclidean distance for the BA model and ER model. Here, the average coupling strength is $\gamma \simeq 0.2816$ for the BA model and $\gamma \simeq 0.3736$ for the ER model. Table Ⅲ and Table Ⅳ shows the average synchronization rate of each network under these conditions.

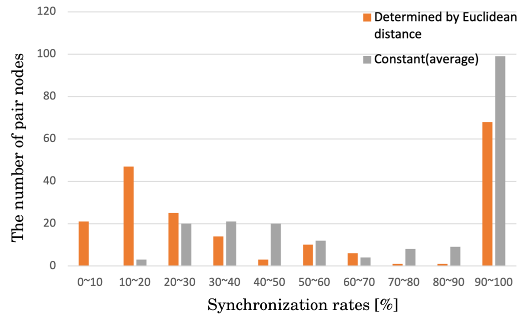


Fig. 7. Relationship between the synchronization rate and the number of pair nodes in the BA model with the coupling strength determined by Euclidean distance and constant (average).

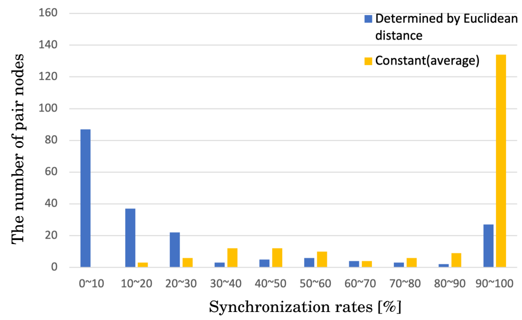


Fig. 8. Relationship between the synchronization rate and the number of pair nodes in the ER model with the coupling strength determined by Euclidean distance and constant (average).

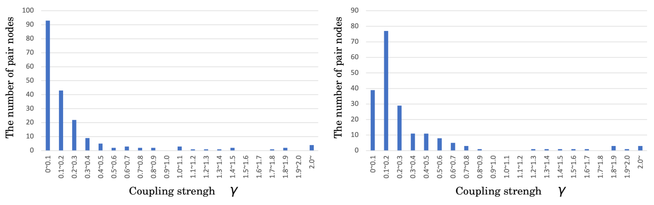
TABLE III
AVERAGE SYNCHRONIZATION RATES FOR DIFFERENT METHODS OF DETERMINING THE COUPLING STRENGTH OF BA MODEL.

Coupling strength	Determinded by distance	Constant (average)
Average synchronization rates[%]	50.811	72.425

TABLE IV
AVERAGE SYNCHRONIZATION RATES FOR DIFFERENT METHODS OF DETERMINING THE COUPLING STRENGTH OF ER MODEL.

Coupling strength	Determinded by distance	Constant (average)
Average synchronization rates[%]	28.346	84.425

Figure 9 shows the relationship between the synchronization rate and the number of pair nodes when the coupling strength is determined by Euclidean distance for (a) BA model and (b) ER model.



(a)BA model (b)ER model

Fig. 9. Coupling strength distribution.

First, we compare the network structure of the two networks used in this study. If we keep the common coupling strength constant, we can see from Fig. 5 and Table I. There is no significant difference in the distribution of the synchronization rate and the average synchronization rate.

As a result, we compare the networks when the coupling strength is determined by Euclidean distance. Figure 6 shows that the BA model has the largest number of node pairs with a synchronization rate of 90-100[%], while the ER model has the largest number of node pairs with a synchronization rate of 0-10[%]. Table Ⅱ also shows that about half of the nodes in the BA model are synchronized, while only about a quarter of the nodes in the ER model are synchronized, indicating that there are significant differences among the networks. In addition, when the coupling strength is kept constant for each network, the distribution of the synchronization rate is higher for both BA model and ER model, as shown in Fig. 7 and Fig. 8. Table Ⅲ and Ⅳ shows that the average synchronization rate is also higher when the coupling strength is constant. This trend is more pronounced in the ER model. The synchronization rate of most pair nodes is 90-100[%] when the coupling strength is constant, as shown in Fig. 8.

Further, Fig. 9 also shows that in both networks, there are many pair nodes for which the distribution of the coupling strength is lower than the average value. Accordingly, when the coupling strength is determined by Euclidean distance, the synchronization rate is much lower than when it is kept constant at the average value.

V. CONCLUSION

In this study, 100 van der Pol oscillators were used to investigate the synchronization phenomena for two networks, the BA model and the ER model. Where the coupling strength was considered as the Euclidean distance between nodes. The results show that the synchronization rate of both networks decreases when Euclidean distance is considered, compared to the constant coupling strength. The decrease was smaller in the BA model than in the ER model. This may be due to the influence of the hub in the BA model.

In the future, we would like to further investigate the effect of hubs and local node pairs such as indirectly coupled pairs.

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