Modified Capon Beamforming Method with Array Interpolation and Spatial Smoothing

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***Abstract*—In this paper, we propose an array interpolation and spatial smoothing based modified Capon beamforming method, where the arbitrary linear array is transformed to a virtual uniform linear array (ULA) by utilizing the interpolation technique, and then the coherency of incident signals can be decorrelated by employing the spatial smoothing preprocessing. Further by increasing the power of array covariance matrix, a modified Capon beamformer is used to estimate the DOAs, where the component corresponding to the signal subspace is suppressed. We analyze the computational complexity of our method with MUSIC method and illustrate the low complexity of our mehtod. The proposed method estimates the direction-of-arrivals (DOAs) of coherent narrowband signals in an arbitrary linear array with high accuracy and low complexity. The effectiveness of the proposed method is verified through numerical examples.**

I. Introduction

In sensor array processing, many DOA estimation methods were proposed in the literature (see, e.g., [1], [2], [3], [4] and references therein), where the subspace-based methods and the beamforming techniques are widely used because of their simplicity. The beamforming is one of the oldest ideas for DOA estimation, and perhaps the most well-known one is the Capon beamformer [5], where the DOAs are estimated by maintaining a constant gain at the incident directions of desired signals and minimizing the power contributed by noise and interference signals. The properties of Capon beamformer was studied in [6], [7], and the Capon method can be applied to the array with arbitrary geometrical configurations and has low computational complexity compared with the subspace-based DOA estimation methods such as the MUSIC [8], where the computationally tremendous eigendecomposition is not required. Further, it was clarified that the resolution of the Capon beamformer can be equal to that of MUSIC when the SNR tends to infinite [9], whereas the performance of Capon beamformer degrades significantly when the signal to noise ratio (SNR) is low. Moreover, the array is usually subject to great uncertainty in practice.

In this paper, we consider the DOA estimation of the coherent narrowband signals impinging on an arbitrary linear array. In [10], we proposed a new computational efficient modified Capon beamforming method and in this paper, we analyze the computational complexity of our mehtod and compared it with MUSIC method[1]. The proposed method overcomes the common restriction of ULA geometry and becomes suitable for more general array geometry than ordinary methods, while the resolution of DOA estimation is improved to be as good as that of the subspace-based methods such as the MUSIC, where the computationally burdensome eigendecomposition is avoided. Finally the simulation results show that the proposed method performs well at low SNR.

II. Problem formulation

We consider an arbitrary linear array composed of M identical and omnidirectional sensors and suppose that p coherent narrowband signals $\{s_i(n)\}_{i=1}^{p}$ with the center frequency f_0 impinging on the array from far-field along the distinct directions $\{\theta_i\}_{i=1}^{p}$. The received noisy array data at the nth snapshot can be expressed as

$$\boldsymbol{y}(n) = \boldsymbol{A}(\theta)\boldsymbol{s}(n) + \boldsymbol{w}(n) \tag{1}$$

where $\boldsymbol{y}(n)$, $\boldsymbol{s}(n)$ and $\boldsymbol{w}(n)$ are the vectors of the received noisy signals, the incident signals, and additive noise respectively given by $\boldsymbol{y}(n) \triangleq [y_1(n), y_2(n), \cdots, y_M(n)]^T$, $\boldsymbol{s}(n) \triangleq [s_1(n), s_2(n), \cdots, s_p(n)]^T$, and $\boldsymbol{w}(n) \triangleq [w_1(n), w_2(n), \cdots, w_M(n)]^T$, while $\boldsymbol{A}(\theta)$ is the array response matrix given by $\boldsymbol{A}(\theta) \triangleq [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \cdots, \boldsymbol{a}(\theta_p)]$, and $(\cdot)^T$ denotes transpose.

Four assumptions need to be made. Firstly, given a set of distinct DOAs $\{\theta_1, \theta_2, \cdots .\theta_p\}$, the array response vectors $\{\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \cdots, \boldsymbol{a}(\theta_P)\}$ are linearly independent. Secondly, the incident signals $\{s_i(n)\}$ are all coherent, which can be expressed as complex multiples of a common signal $s_1(n)$ as $s_i(n) = \beta_i s_1(n)$, where β_i is a complex attenuation coefficient of ith signal and the signal $s_1(n)$ is temporally complex Gaussian random process with zero-mean. The variance of signal $s_1(n)$ is given by $E\{s_1(n)s_1^*(t)\} = r_s\delta_{n,t}$, $E\{s_1(n)s_1(t)\} = 0, \forall n, t$ where $E\{\cdot\}$, $(\cdot)^*$, and $\delta_{n,t}$ denote the expectation, the complex conjugate, and the Kronecker delta respectively. Thirdly, the additive noises $\{w_i(n)\}$ are temporally and spatially complex white Gaussian random process with zero-mean and variance σ^2, where $E\{\boldsymbol{w}(n)\boldsymbol{w}^H(t)\} = \sigma^2\boldsymbol{I}_M\delta_{n,t}$, $E\{\boldsymbol{w}(n)\boldsymbol{w}^T(t)\} = \boldsymbol{O}_{M\times M}$,

[1]The paper includes the contents presented in [10].

while $\boldsymbol{I}_m$, $\boldsymbol{O}_{m\times q}$, and $(\cdot)^H$ indicate the $m \times m$ identity matrix, the $m \times q$ null matrix, and the Hermitian transpose. Additionally the additive noises are uncorrelated with the incident signals; Lastly, the number of incident signals p is known and it satisfies the relation $M \geq 1.5p$.

This paper concentrates on estimating the DOAs $\{\theta_i\}_{i=1}^{p}$ of coherent signals impinging on an arbitrary linear array from the noisy array data $\{\boldsymbol{y}(n)\}_{n=1}^{N}$ in a computationally efficient way.

III. Proposed Method

A. Interpolation Transformation

We design virtual array as uniformly spaced linear array to make sure the effective implementation of spatial smoothing preprocessing and polynomial rooting. Firstly we divide the array scanning area into K sectors and implement interpolation transformation on each sector separately, where the kth sector is defined by the interval $[\theta_k^{(1)}, \theta_k^{(2)}]$, and we define a set of angles $\boldsymbol{\Theta}_k$ for each sector with a fixed interval $\Delta\theta$ [11], [12], then we can get the response matrix $\boldsymbol{A}_k$ of actual array and $\bar{\boldsymbol{A}}_k$ of virtual ULA associated with the set $\boldsymbol{\Theta}_k$ respectively. There exists a constant matrix $\boldsymbol{B}_k$ which satisfies that

$$\boldsymbol{B}_k \boldsymbol{A}_k = \bar{\boldsymbol{A}}_k \tag{2}$$

where the interpolated matrix $\boldsymbol{B}_k$ can be estimated as the least squares solution of (2), and the interpolation error is defined as $\varepsilon \triangleq (\|\bar{\boldsymbol{A}}_k - \boldsymbol{B}_k\boldsymbol{A}_k\|)/(\|\bar{\boldsymbol{A}}_k\|)$. According to experience, we hope that the difference between the position of the real array and that of the virtual array is as small as possible [12]. By increasing the number of divided sectors, the interpolation error decreases, while the computational burden becomes heavier. Typically we choose the minimum in the set of K, which meet the condition that $\varepsilon < 10^{-3}$ as the number of sectors [12]. Then compute $\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots \boldsymbol{B}_k$ in each sector separately. For notational convenience, hereafter we use $\boldsymbol{B}$ to represent $\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots \boldsymbol{B}_k$. From (2), we easily get the covariance matrix $\bar{\boldsymbol{R}}$ of the virtual ULA as

$$\bar{\boldsymbol{R}} = \boldsymbol{B}\boldsymbol{R}\boldsymbol{B}^H = \bar{\boldsymbol{A}}\boldsymbol{R}_s\bar{\boldsymbol{A}}^H + \sigma^2\boldsymbol{B}\boldsymbol{B}^H \tag{3}$$

Here the white noise is converted into colored noise, so a prewhitening is necessary. By partitioning the convariance matrix $\boldsymbol{R}$ as

$$\boldsymbol{R} = \begin{array}{cc} \begin{array}{cc} \scriptstyle p & \scriptstyle M-p \end{array} & \\ \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22} \end{bmatrix} & \begin{array}{l} \scriptstyle p \\ \scriptstyle M-p \end{array} \end{array} \tag{4}$$

we get the estimate of noise variance σ^2 as [14]

$$\sigma^2 = \frac{\mathrm{tr}\{\boldsymbol{R}_{22}\boldsymbol{\Pi}\}}{\mathrm{tr}\{\boldsymbol{\Pi}\}} \tag{5}$$

where $\boldsymbol{\Pi} = \boldsymbol{I}_{M-p} - \boldsymbol{R}_{21}\boldsymbol{R}_{21}^{\dagger}$, and $\boldsymbol{R}_{21}^{\dagger} = (\boldsymbol{R}_{21}^{H}\boldsymbol{R}_{21})^{-1} \cdot\boldsymbol{R}_{21}^{H}$. Clearly by substituting (5) into (3), the 'prewhitened' covariance matrix of the virtual array $\tilde{\mathbf{R}}$ is given by:

$$\tilde{\boldsymbol{R}} \triangleq \bar{\boldsymbol{A}}\boldsymbol{R}_s\bar{\boldsymbol{A}}^H + \sigma^2\boldsymbol{I}_M = \bar{\boldsymbol{R}} - \sigma^2\boldsymbol{B}\boldsymbol{B}^H + \sigma^2\boldsymbol{I}_M \tag{6}$$

By using received data from the real array, we obtain the covariance matrix $\tilde{\boldsymbol{R}}$ of the virtual array when same incident signals imping on the virtual array. Hence the given arbitrary array is converted into virtual ULA successfully.

B. FBSS Preprocessing

The coherence between the incident signals will lead to the rank deficit of covariance matrix $\tilde{\mathbf{R}}$ in (6) So we use spatial smoothing of subarrays to decorrelate incident signals here. We divide the virtual uniform linear array into L overlapping subarrays with M_0 ($M_0 \geq p+1$) sensors in the forward and backward directions, where $L = M - M_0 + 1$, and sensors $\{l, l+1, \cdots, l+M_0-1\}$ or $\{M-l+1, M-l, \cdots, L-l+1\}$ compose the lth forward subarray or backward subarray respectively, where $l = 1, 2, \cdots, L$. From (6), the covariance matrixs of the lth forward subarray and the lth backward subarray are respectively given by:

$$\tilde{\boldsymbol{R}}_l^f = \boldsymbol{A}_1\boldsymbol{D}^{l-1}\boldsymbol{R}_s\boldsymbol{D}^{-(l-1)}\boldsymbol{A}_1^H + \sigma^2\boldsymbol{I}_{M_0} \tag{7}$$

and

$$\tilde{\boldsymbol{R}}_l^b = \boldsymbol{A}_1\boldsymbol{D}^{-(M_0+l-2)}\boldsymbol{R}_s^*\boldsymbol{D}^{M_0+l-2}\boldsymbol{A}_1^H + \sigma^2\boldsymbol{I}_{M_0} \tag{8}$$

where $\boldsymbol{A}_1$ is the first M_0 rows of response matrix $\bar{\boldsymbol{A}}$ and $\boldsymbol{D}$ denotes the $p \times p$ diagonal matrix

$$\boldsymbol{D} \triangleq \mathrm{diag}\left(e^{-j(2\pi f_0 d/c)\sin(\theta_1)}, \cdots, e^{-j(2\pi f_0 d/c)\sin(\theta_p)}\right) \tag{9}$$

then we easily obtain the forward spatially smoothed covariance matrix and the backward spatially smoothed covariance matrix as $\tilde{\boldsymbol{R}}^f = \frac{1}{L}\sum\limits_{l=1}^{L}\tilde{\boldsymbol{R}}_l^f$ and $\tilde{\boldsymbol{R}}^b = \frac{1}{L}\sum\limits_{l=1}^{L}\tilde{\boldsymbol{R}}_l^b$.

In order to minimize the loss of effective aperture, we can obtain a interpolated spatially forward-backward smoothed covariance matrix $\tilde{\boldsymbol{R}}_{\mathrm{FB}}$ as

$$\tilde{\boldsymbol{R}}_{\mathrm{FB}} = \frac{1}{2}\left(\tilde{\boldsymbol{R}}^f + \tilde{\boldsymbol{R}}^b\right) \tag{10}$$

It has been proved that spatially smoothed covariance matrix $\tilde{\boldsymbol{R}}_{\mathrm{FB}}$ is nonsingular, when number of subarrays is greater than or equal to an half of the number of signals, i.e., $L \geq 0.5p$ [13].

C. Modified Capon Beamformer

Based on the spatially smoothed covariance matrix $\tilde{\boldsymbol{R}}_{\mathrm{FB}}$ in (10), the standard Capon beamformer (SCB) for the DOA estimation of the coherent signals is given by

$$\min_{\bar{\boldsymbol{w}}} \bar{\boldsymbol{w}}^H\tilde{\boldsymbol{R}}_{\mathrm{FB}}\bar{\boldsymbol{w}} \qquad s.t. \qquad \bar{\boldsymbol{w}}^H\boldsymbol{a}(\theta) = 1 \tag{11}$$

where the weight vector $\bar{\boldsymbol{w}}$ is obtained by

$$\bar{\boldsymbol{w}}_{\mathrm{SCB}} = \frac{\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-1}\bar{\boldsymbol{a}}(\theta)}{\bar{\boldsymbol{a}}(\theta)^H\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-1}\bar{\boldsymbol{a}}(\theta)} \tag{12}$$

Then the DOAs of incident signals can be estimated from the peaks of spatial spectrum $P_{\mathrm{SCB}}(\theta)$ or the valleys of cost function $f_{\mathrm{SCB}}(\theta)$

$$P_{\mathrm{SCB}}(\theta) \triangleq \frac{1}{\boldsymbol{a}^H(\theta)\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-1}\boldsymbol{a}(\theta)} = \frac{1}{f_{\mathrm{SCB}}(\theta)} \tag{13}$$

By replacing $\tilde{\boldsymbol{R}}_{\mathrm{FB}}$ in the SCB constraint function with $\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{m}$, where m is a positive integer (i.e., $m \geq 1$), we can obtain a

MCB to design the optimal weight vector $\bar{\boldsymbol{w}}$ by solving the following problem

$$\min_{\bar{\boldsymbol{w}}} \bar{\boldsymbol{w}}^H \tilde{\boldsymbol{R}}_{\mathrm{FB}}^m \bar{\boldsymbol{w}} \quad \text{subject to} \quad \bar{\boldsymbol{w}}^H \boldsymbol{a}(\theta) = 1. \tag{14}$$

Similarly the solution to the weight vector and the MCB spatial spectrum are given by

$$\bar{\boldsymbol{w}}_{\mathrm{MCB}} = \frac{\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-m} \boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta)\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-m} \boldsymbol{a}(\theta)} \tag{15}$$

$$P_{\mathrm{MCB}}(\theta) = \frac{1}{f_{\mathrm{MCB}}(\theta)} \tag{16}$$

where $f_{\mathrm{MCB}}(\theta)$ is the MCB cost function defined by

$$f_{\mathrm{MCB}}(\theta) \triangleq \boldsymbol{a}^H(\theta)\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-m} \boldsymbol{a}(\theta). \tag{17}$$

Apparently the DOAs $\{\theta_k\}_{k=1}^{p}$ can be estimated by maximizing the spectrum $P_{\mathrm{MCB}}(\theta)$ in (16) without the procedure of eigendecomposition. Noted that the proposed MCB in (16) reduces to the SCB in (13) when $m = 1$.

Benefiting from the Vandermonde structure of the response matrix $\bar{\boldsymbol{A}}$ of the virtual array, we use a polynomial rooting to replace the peaks searching here. By defining a complex variable z as $z = e^{j\omega}$, where $\omega = 2\pi f_0 d \sin\theta / c$, from (17), we get the z format cost function and the corresponding relation between DOA and complex variable z as:

$$f_{\mathrm{MCB}}(\theta) = f_{\mathrm{root-MCB}}(z) = \bar{\boldsymbol{a}}^H(z)\tilde{\boldsymbol{R}}_{\mathrm{FB}}^{-m} \bar{\boldsymbol{a}}(z) \tag{18}$$

$$\sin\theta_i = \frac{\arg(z_i)c}{2\pi f_0 d} \tag{19}$$

where $\bar{\boldsymbol{a}}(\theta) = \bar{\boldsymbol{a}}(z) = [1, z, \cdots, z^{M-1}]$. Finally, we obtain the DOAs of incident signals by finding the minimum values of $f_{\mathrm{root-MCB}}(z)$.

IV. Complexity Analysis

The computational complexity of Capon method is $F_{Capon} = 10NM^2 + F_{inv}(M) + (8M(M+1)+1)F(r)$, where $F(r)$ is a constant which is only decided by the angular resolution r, and $F_{inv}(M)$ represents the complexity of inversion operation of matrix with M dimensions. While, the computation complexity of MUSIC method is $F_{MUSIC} = 10NM^2 + F_{evd}(M) + F_{MDL}(M) + (8(M-N)(M+1)+1)F(r)$, where $F_{evd}(M)$ represents the complexity of eigendecomposition operation of matrix with M dimensions, and $F_{MDL}(M)$ is the MDL criterion which has an approximate complexity of $\mathrm{O}\left(M^2\right)$. Due to the high complexity of matrix eigendecomposition and matrix inversion, the complexity of Capon method and MUSIC method actually is decided by the calculation of eigendecomposition and inversion of covariance matrix respectively. At the same time, interpolated transformation has nearly the same influnce on Interpolated MUSIC method and our proposed method. If we set M equals to 10, then the average computation of eigendecomposition operation is 129215 flops, while that of inversion operation is 10736 flops. We can see the complexity of eigendecomposition is over 10 times than that of inversion, and their difference become even bigger with the increase of M.

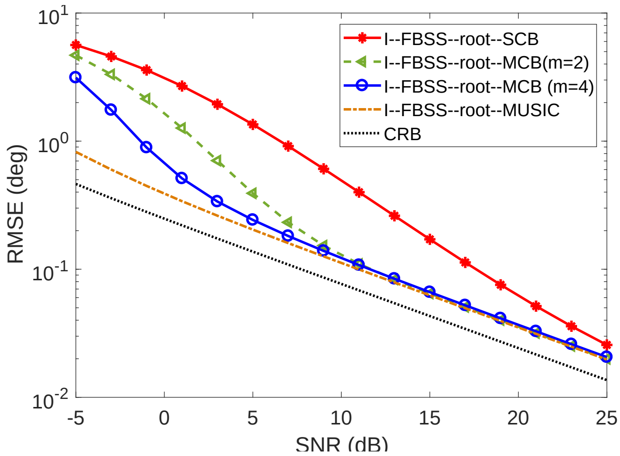

Fig. 1. Estimation performance versus SNR

V. Numerical Results

In this section, we evaluate effectiveness of the proposed method (I-FBSS-root-MCB) for DOA estimation of coherent narrowband signals by using an arbitrary linear array, which is constructed by adding a horizontal shift Δd_i to the ith ($i = 1, 2, \cdots M$) element of a virtual ULA consisting of $M = 9$ sensors with element spacing $d = \lambda/2$. Here, we set the horizontal shift vector as

$$\begin{aligned}\boldsymbol{\Delta_d} = [&0, 0, 0.1\lambda, -0.1\lambda, 0.05\lambda,\\ &-0.05\lambda, 0.1\lambda, -0.05\lambda, -0.05\lambda]^T.\end{aligned} \tag{20}$$

Two coherent signals with equal power arrive from angles $\theta_1{=}10^o$ and $\theta_2{=}\,25^o$. The number of snapshots is set as $N = 500$. The RMSEs of estimates of the DOAs versus SNRs are shown in Fig. 1.

For comparison, the behavior of the FBSS-based standard Capon beamformer with interpolation (I-FBSS-root-SCB), the FBSS-based root-MUSIC with interpolation (I-FBSS-root-MUSIC) [11], and the Cramer-Rao lower bound (CRB) [15] are also carried out. The results shown in Fig. 1 are all based on 1000 independent trials. The empirical root-mean-square error (RMSE) in the simulations is calculated by $\mathrm{RMSE} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{P}\left(\theta_i - \hat{\theta}_i^{(t)}\right)^2}$, where T denotes the total number of trials and $\hat{\theta}_i^{(t)}$ is the estimate of θ_i obtained in the tth trial. In addition, we choose $m = \{2, 4\}$ for the experiments.

From the curves in Fig. 1, we can see that all methods estimate the DOAs of the coherent signals relatively accurately and the RMSEs decrease significantly with the increase of SNR, benefiting from the utilizing of interpolation transform and FBSS preprocessing. Meanwhile, the proposed method completely outperforms the I-FBSS-root-SCB method, and its performance becomes better with the increase of power m. When m reaches 4, the resolution of proposed method is very close to the I-FBSS-root-MUSIC method, which is known for its precision.

VI. Conclusion

In this paper, we propose an array interpolation and spatial smoothing based modified Capon beamforming method for DOA estimation of coherent signals with arbitrary linear array. The simulation results show that the proposed method performs as well as the subspace-based method, where the computationally burdensome eigendecomposition is avoided. Meanwhile, the proposed method overcomes the common restriction of ULA geometry and becomes suitable for more general array geometry than ordinary methods.

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