Theoretical Approach of Oscillation Death in Polygonal Oscillatory Networks

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Abstract—We have observed oscillation death in two coupled polygonal oscillatory networks with strong frustration. In order to understand the mechanism of these phenomena, in this study, theoretical analysis using power consumption is applied to solve the amplitude of the coupled oscillators.

I. INTRODUCTION

In recent years, we have focused on synchronization phenomena of coupled oscillators under a difficult situation for the circuit. Setou et al. have reported the synchronization phenomena in \( N \) oscillators coupled by resistors as a ring. The oscillation stop in some range of the coupling resistors was confirmed [1].

We have investigated the synchronization phenomena in the coupled polygonal oscillatory networks sharing branches [2]. In this system, van der Pol oscillators are connected to every corner of polygonal network. By using computer simulations and theoretical analysis, we confirm that the coupled oscillators tend to synchronize to minimize the power consumption of the whole system. The phase difference of the shared oscillators is solved by finding the minimum value of the power consumption function.

Recently, in order to investigate amplitude change of the oscillators with strong coupling parameter, we have proposed a new circuit model of coupled polygonal network which is inserted the earth resistances in all ground parts [3], [4]. In this circuit system, we confirm that the amplitude of the oscillators decreases by increasing the value of the coupling strength and oscillation death of the oscillators located farthest place from the shared oscillators is occurred.

In order to make clear the mechanism of these phenomena especially oscillation death, in this study, theoretical analysis using power consumption of the coupling resistance is applied to solve the amplitude of the coupled oscillators.

II. CIRCUIT MODEL

A. Two Coupled Oscillatory Networks (Symmetric Case)

Two identical polygonal oscillatory networks are coupled by sharing a branch as shown in Fig. 1. In this circuit model, we consider the coupling method which two adjacent oscillators are tend to synchronize at anti-phase state. We call the first and the second oscillators which are connected to both side of polygonal network “shared oscillators.” Figure 1 (b) shows the circuit model of the 5-5 coupling network.

Next, we develop the expression for the circuit equations of \( N-N \) coupling oscillatory networks as shown in Fig. 1. The \( v_k = i_{R_k} \) characteristics of the nonlinear resistor are approximated by the following third order polynomial equation,

\[
i_{R_k} = -g_1 v_k + g_3 y_k^3 \quad (g_1, g_3 > 0), (k = 1, 2, 3, 4). \tag{1}\n\]

The normalized circuit equations governing the circuit are expressed as [kth oscillator]

\[
\begin{align*}
\frac{dx_k}{d\tau} &= \varepsilon \left( 1 - \frac{1}{3} x_k^2 \right) x_k - (y_{ak} + y_{bk} + y_{ck}) \\
\frac{dy_{ak}}{d\tau} &= \frac{1}{3} \left( x_k - \eta y_{ak} - \gamma (y_{ak} + y_{bn}) \right) \\
\frac{dy_{bk}}{d\tau} &= \frac{1}{3} \left( x_k - \eta y_{bk} - \gamma (y_{bk} + y_{bn}) \right) \\
\frac{dy_{ck}}{d\tau} &= \frac{1}{3} \left( x_k - \eta y_{ck} - \gamma (y_{ck} + y_{bn}) \right) \\
&\quad (k = 1, 2, 3, 4). \tag{2}\n\end{align*}
\]

In this equations, \( \gamma \) is the coupling strength, \( \varepsilon \) denotes the
nonlinearity of the oscillators and \( y_n \) denotes the current of neighbor oscillator on coupling resistor.

**B. Oscillation Death**

Figure 2 shows the observed attractors of 5 - 5 coupling network by changing the coupling strength. In this circuit model, the amplitude of fourth oscillator (which is located farthest place from the shared oscillators) decreases with the coupling strength. Then, we observe oscillation death of the fourth oscillator when the coupling strength is set to \( \gamma = 1.0 \).

![Attractor (5-5 coupling network).](image)

Figure 3 shows the change of the amplitude observed from 5-5, 7-7, 9-9 and 11-11 coupling networks. From these figures, we can see that first, the oscillation death of the oscillators located farthest place from the shared oscillators is occurred. After that, the other oscillators stop to oscillate at same time.

We explain the oscillation death as physical phenomena. The earth resistance is not inserted in two shared oscillators, namely the shared oscillators do not tend to oscillation death by controlling the phase difference to minimize energy. Then next oscillators from the shared oscillators try to oscillate to synchronize with the shared oscillators. Furthermore, after oscillation death is occurred, the network topology is changed from ring to two ladders.

Why only one oscillator of one side polygonal network stops their oscillation? Figure 4 summarizes possibility of the network structures after oscillation death for 5-5 coupling network. We compare the power consumption of each pattern by using the amplitude which is obtained at oscillation death. From this figure, we can see that the power consumption of pattern A is smaller than pattern B. Namely, 5-5 coupling network changes the network structure to pattern A, after oscillation death. Tables I-IV show the power consumption for each network pattern of other coupling networks. From these tables, the power consumption shows lowest value when only one oscillator of one side polygonal network stops their oscillation for any kinds of the coupling networks.

**C. Power Consumption**

How is the amplitude of the coupled oscillators determined? In order to make clear the mechanism of the oscillation death, we focus on the power consumption of the coupling resistors in the whole system.

We assume the current of the inductor \((3L)\) as the following equation.

\[
i_k(t) = \rho_k \sin(\omega t + \varphi_k) \quad (k = 1, 2, 3...O_N),
\]

where \( \omega \) is the natural frequency of van der Pol oscillator as \( \omega = \frac{1}{\sqrt{LC}} \). When the coupling resistance is assumed as \( R = 1 \), the average power consumption of the coupling resistor
between \(k\)th and \((k+1)\)st oscillators are described as Eq. (4).

\[
P = \frac{1}{2\pi} \int_0^{2\pi} \left( i_k(t) + i_{k+1}(t) \right)^2 dt \tag{4}
\]

Here, we try to derive each amplitude theoretically with the following assumptions. First, let the phase difference between the oscillators be \(\pi\). Second, the amplitude of each oscillator has different value. So, we fix each amplitude depending on the located place of the oscillators. Figure 5 (a) shows the amplitude setting for 5-5 coupling network as one example. The amplitude of the shared oscillators is set to \(\rho_0\) and the next oscillators of the shared oscillators are set to \(\rho_1\) and \(\rho_2\).

By using Eq. (4), we can obtain the power consumption of the whole systems for 5-5 coupling network as follows.

\[
P = 4(\rho_0 - \rho_1)^2 + 4(\rho_1 - \rho_2)^2. \tag{5}
\]

We calculate the power consumption of the whole system by using the amplitude \((\rho_0\) and \(\rho_2)\) obtained from the computer simulations. Figure 6 (a) shows the theoretical results of the amplitude \((\rho_1)\) of 5-5 coupling networks. The results of the theoretical analysis match well with simulation results.

The amplitude of other types of the coupling networks is also solved by using the following equations.

- **7-7 coupling network:**

\[
P = 4(\rho_0 - \rho_1)^2 + 4(\rho_1 - \rho_2)^2 + 4(\rho_2 - \rho_3)^2. \tag{6}
\]
• 9-9 coupling network:
  \[ P = 4(\rho_0 - \rho_1)^2 + 4(\rho_1 - \rho_2)^2 + 4(\rho_2 - \rho_3)^2 + 4(\rho_0 - \rho_3)^2. \] (7)

• 11-11 coupling network:
  \[ P = 4(\rho_0 - \rho_1)^2 + 4(\rho_1 - \rho_2)^2 + 4(\rho_2 - \rho_3)^2 + 4(\rho_3 - \rho_4)^2. \] (8)

The results are shown in Fig. 6 (b), (c) and (d). From these results, we can see that the theoretical analysis can express the characteristics of the amplitude change.

III. CONCLUSIONS

In this study, we have investigated synchronization phenomena in coupled polygonal oscillatory networks with strong frustrations. We focused on the amplitude of each oscillator when the coupling strength is increased. By using the computer simulations, we confirmed that the amplitude of the oscillators decreases by increasing the coupling strength and oscillation death is occurred at un-frustrated oscillators. Furthermore, we observed the amplitude change in the asymmetric network model.

ACKNOWLEDGMENT

This work was partly supported by JSPS Grant-in-Aid for Challenging Exploratory Research 26540127.

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