Synchronization Phenomenon in Four Chains of Coupled Oscillators

Kosuke Matsumura, Takahiro Nagai, Yoko Uwate and Yoshifumi Nishio
Dept. Electrical and Electronic Eng, Tokushima University, Tokushima 770-8506, JAPAN
Email: {kosuke, nagataka, uwate, nishio}@ee.tokushima-u.ac.jp

***Abstract*— In this study, we investigate a synchronization phenomenon from a circuit network which is composed of two kinds of oscillator chains. They are one-dimensional arrays of weakly coupled van der Pol oscillators. From computer simulations, we observe relatively interesting unexpected synchronization phenomenon.**

I. Introduction

Synchronization phenomena are very basic phenomena. Also, the important phenomena observed everywhere in natures. For example, vibration of a pendulum, firefly luminescence, gate patterns of four-leg animals, two frogs using voice religiously, periodic swinging of candle flames, and so on. Therefore, coupled oscillators are good models to investigate such interesting synchronization phenomena. Many researchers have proposed different coupled oscillatory networks and have discovered many interesting synchronization phenomena [1]-[8]. The research group of the authors is also working on coupled oscillatory networks. Especially, we have been interested in coupled oscillators whose connections cause some kinds of frustrations [9]-[12].

In this study, we investigate a circuit network which is composed of two kinds of oscillator chains. They are one-dimensional arrays of weakly coupled van der Pol oscillators. In this network, we couple the oscillators at bottom four oscillators in the chains to constrain them to produce in-phase synchronizations. While we couple the oscillators at the top four oscillators in the chains to produce anti-phase synchronizations. Middle oscillators in the chains are not coupled with the other chains. We observe relatively interesting unexpected synchronization phenomenon from computer simulations.

II. Circuit Model

Figure 1 shows the our proposed circuit model. Each oscillator-chain consists of n van der Pol oscillators weakly coupled by resistors r. In the figure, bottom four oscillators are coupled by relatively strong resistors R_i, while the top four oscillators are coupled by also relatively strong resistors R_a via inductors. Middle oscillators are not coupled with oscillators located in the horizontal direction but weakly coupled vertically.

The coupling structure on the bottom or in vertical couplings (by R_i or r) tends to make the oscillators to synchronize in in-phase. While the coupling structure on the top (by R_a via inductor) tends to make the oscillators to synchronize in anti-phase.

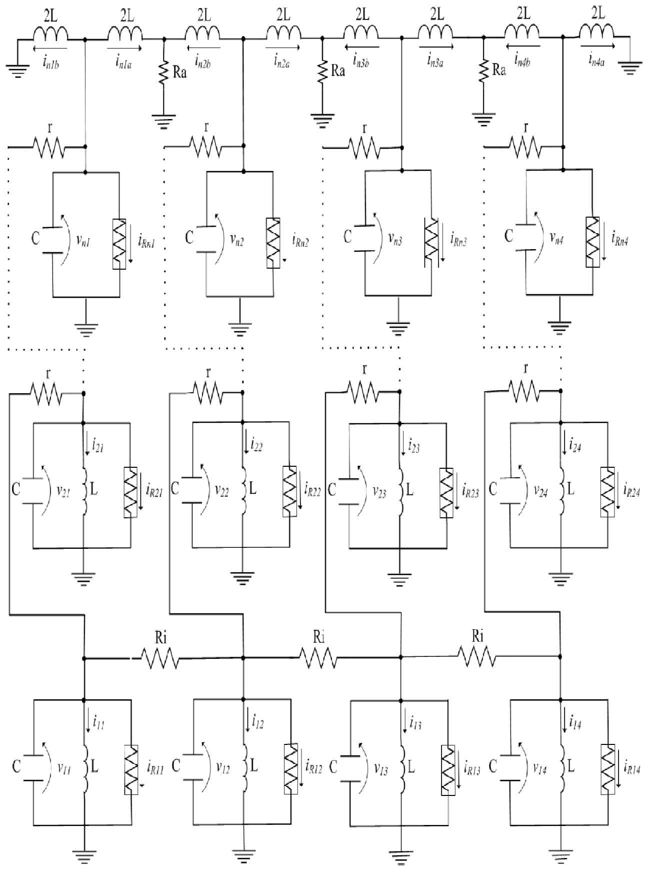

Fig. 1. Circuit model for the case of n oscillator-chains.

We define the bottom four oscillators as Osc_{11}, Osc_{12}, Osc_{13} and Osc_{14} from the left, those on the kth row from the bottom as Osc_{k1}, Osc_{k2}, Osc_{k3} and Osc_{k4}, and the top four oscillators as Osc_{n1}, Osc_{n2}, Osc_{n3} and Osc_{n4}.

First, we assume that the $v-i$ characteristics of the nonlinear resistor in each oscillator is given by the following third order polynomial equation.

$$i_{R_{kj}} = -g_1 v_{kj} + g_3 {v_{kj}}^3 \tag{1}$$

where $g_1, g_3 > 0$, $k = 1, 2, ..., n$, and $j = 1, 2, 3, 4$.

By using the following variables and parameters:

$$t = \sqrt{LC}\tau,\ v_{kj} = \sqrt{\frac{g_1}{g_3}}\, x_{kj},\ i_{kj} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{kj},$$

$$\varepsilon = \frac{1}{g_1}\sqrt{\frac{L}{C}},\ \alpha_i = \frac{1}{R_i}\sqrt{\frac{L}{C}},\ \alpha_a = R_a\sqrt{\frac{L}{C}},\ \beta = \frac{1}{r}\sqrt{\frac{L}{C}},$$

the normalized circuit equations are given as follows:
(1) Top oscillators:

$$\left\{\begin{array}{l}
\dot{x}_{n1} = \varepsilon(1 - x_{n1}{}^2)x_{n1} - (y_{n1a} + y_{n1b}) + \beta(x_{n1} - x_{(n-1)1}) \\
\dot{y}_{n1a} = 0.5\{x_{n1} - \alpha_a(y_{n1a} + y_{n2b})\} \\
\dot{y}_{n1b} = 0.5\, x_{n1} \\
\dot{x}_{n2} = \varepsilon(1 - x_{n2}{}^2)x_{n2} - (y_{n2a} + y_{n2b}) + \beta(x_{n2} - x_{(n-1)2}) \\
\dot{y}_{n2a} = 0.5\{x_{n2} - \alpha_a(y_{n2a} + y_{n3b})\} \\
\dot{y}_{n2b} = 0.5\{x_{n2} - \alpha_a(y_{n1a} + y_{n2b})\} \\
\dot{x}_{n3} = \varepsilon(1 - x_{n3}{}^2)x_{n3} - (y_{n3a} + y_{n3b}) + \beta(x_{n3} - x_{(n-1)3}) \\
\dot{y}_{n3a} = 0.5\{x_{n3} - \alpha_a(y_{n3a} + y_{n4b})\} \\
\dot{y}_{n3b} = 0.5\{x_{n3} - \alpha_a(y_{n2a} + y_{n3b})\} \\
\dot{x}_{n4} = \varepsilon(1 - x_{n4}{}^2)x_{n4} - (y_{n4a} + y_{n4b}) + \beta(x_{n4} - x_{(n-1)4}) \\
\dot{y}_{n4a} = 0.5\, x_{n4} \\
\dot{y}_{n4b} = 0.5\{x_{n4} - \alpha_a(y_{n3a} + y_{n4b})\}
\end{array}\right. \tag{2}$$

(2) Middle oscillators ($k = 2, 3, \cdots, n-1$) :

$$\left\{\begin{array}{l}
\dot{x}_{k1} = \varepsilon(1 - x_{k1}{}^2)x_{k1} - y_{k1} + \beta(x_{(k+1)1} - 2x_{k1} + x_{(k-1)1}) \\
\dot{y}_{k1} = x_{k1} \\
\dot{x}_{k2} = \varepsilon(1 - x_{k2}{}^2)x_{k2} - y_{k2} + \beta(x_{(k+1)2} - 2x_{k2} + x_{(k-1)2}) \\
\dot{y}_{k2} = x_{k2} \\
\dot{x}_{k3} = \varepsilon(1 - x_{k3}{}^2)x_{k3} - y_{k3} + \beta(x_{(k+1)3} - 2x_{k3} + x_{(k-1)3}) \\
\dot{y}_{k3} = x_{k3} \\
\dot{x}_{k4} = \varepsilon(1 - x_{k4}{}^2)x_{k4} - y_{k4} + \beta(x_{(k+1)4} - 2x_{k4} + x_{(k-1)4}) \\
\dot{y}_{k4} = x_{k4}
\end{array}\right. \tag{3}$$

(3) Bottom oscillators:

$$\left\{\begin{array}{l}
\dot{x}_{11} = \varepsilon(1 - x_{11}{}^2)x_{11} - y_{11} + \beta(x_{21} - x_{11}) - \alpha_i(x_{11} - x_{12}) \\
\dot{y}_{11} = x_{11} \\
\dot{x}_{12} = \varepsilon(1 - x_{12}{}^2)x_{12} - y_{12} + \beta(x_{22} - x_{12}) \\
\qquad\qquad + \alpha_i(x_{11} - 2x_{12} + x_{13}) \\
\dot{y}_{12} = x_{12} \\
\dot{x}_{13} = \varepsilon(1 - x_{13}{}^2)x_{13} - y_{13} + \beta(x_{23} - x_{13}) \\
\qquad\qquad + \alpha_i(x_{12} - 2x_{13} + x_{14}) \\
\dot{y}_{13} = x_{13} \\
\dot{x}_{14} = \varepsilon(1 - x_{14}{}^2)x_{14} - y_{14} + \beta(x_{24} - x_{14}) - \alpha_i(x_{14} - x_{13}) \\
\dot{y}_{14} = x_{14}
\end{array}\right. \tag{4}$$

where x_{kj} corresponds to the voltage across the capacitor and y_{kj}, y_{nja}, y_{njb} are the currents through the inductors of Osc_{kj}.

III. Synchronization Phenomena

Figures 2 and 3 show the computer simulation results for the case of $n = 3$. Also, figures 4 and 5 show results for the case of $n = 4$. The circuit equations (2)-(4) are calculated by using the fourth-order Runge-Kutta method with the step size $h = 0.005$. The circuit parameters are chosen as $\varepsilon = 0.10$, $\alpha_i = \alpha_a = 0.5$, and $\beta = 0.02$.

As we expected, the bottom four oscillators (Osc_{11}, Osc_{12}, Osc_{13} and Osc_{14}) are synchronized in in-phase and the top four oscillators (Osc_{31}, Osc_{32}, Osc_{33} and Osc_{34}) are synchronized in anti-phase.

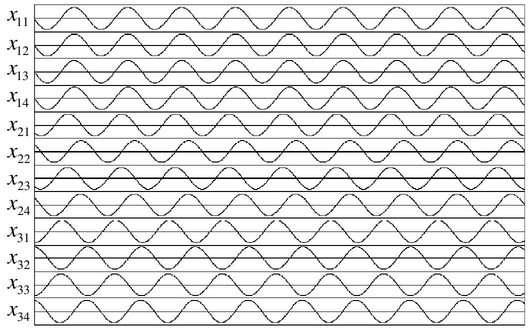

Fig. 2. Computer simulation results (time waveform) for $n = 3$.

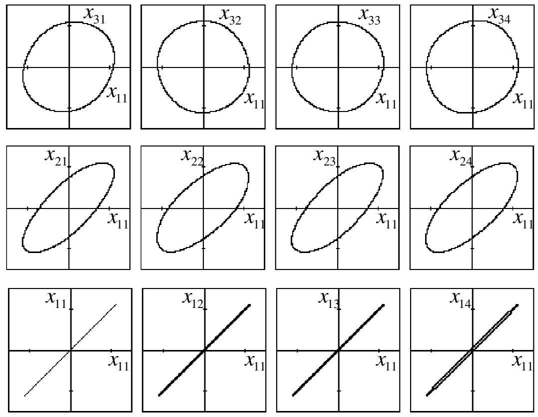

Fig. 3. Computer simulation results (phase shift) for $n = 3$.

However, we observe unexpected synchronization phenomenon between the oscillators in each chain. Because the oscillators in each chain are coupled by r vertically, we expected that the middle four oscillators (Osc_{21}, Osc_{22}, Osc_{23}, Osc_{24},) are quasi-synchronization. And, the top four oscillators (Osc_{31}, Osc_{32}, Osc_{33} and Osc_{34}) are which phase difference is around 90[*deg*.]. Also, we can observe from Figs. 2 and 3 that the oscillators in the middle row (Osc_{21}, Osc_{22}, Osc_{23} and Osc_{24}) have some amount of phase shift to the first row, and those in the top row (Osc_{31}, Osc_{32}, Osc_{33} and Osc_{34}) show larger phase shift.

IV. Conclusions

In this study, we have observed a synchronization phenomenon from a circuit network which is composed of two kinds of oscillator chains. Their edges were coupled to constrain their phase states to generate a frustration. By computer simulations, we could observe interesting unexpected synchronization phenomenon.

The results in this study would be a good model of various natural and artificial systems. For example, stone-paved square of an old town (e.g. we observed one in Evora, Portugal) starts from one edge of the square to line up stones in a regular way like in-phase. However, sometimes the other edge of the square has a different constraint like anti-phase. In that case, frustrations occur somewhere in the square and they are not

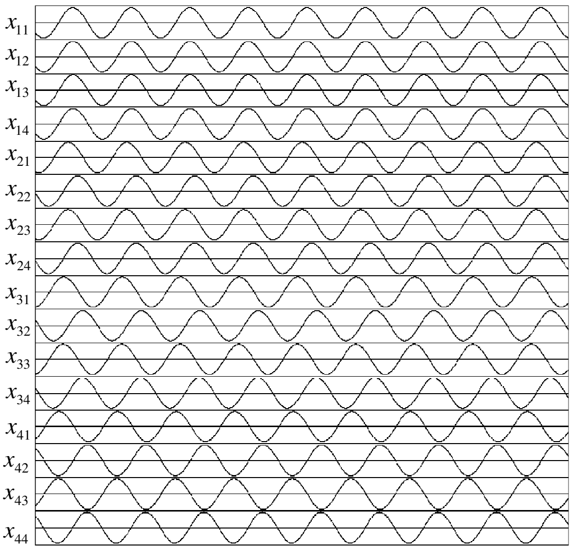

Fig. 4. Computer simulation results (time waveform) for $n = 4$.

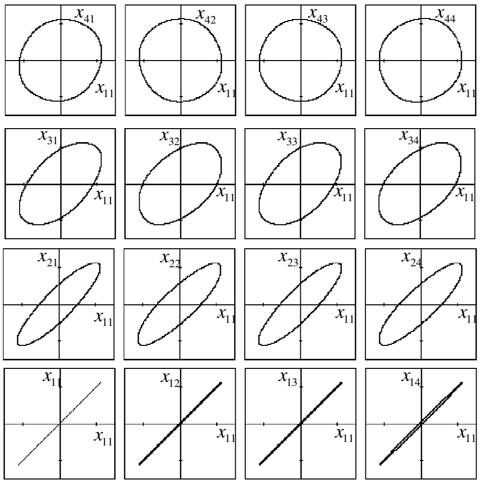

Fig. 5. Computer simulation results (phase shift) for $n = 4$.

compensated at one or some particular points but over a wide area of the square. That is one example of synchronization phenomena observed from oscillator networks with frustration.

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