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# Synchronization Phenomena of Four Oscillators Coupled as a Regular Tetrahedron Form

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***Abstract*— In our study, we research the synchronization phenomena of four oscillators coupled as a regular tetrahedron form. We connect four van der Pol oscillators in the tetrahedron form and survey the phase difference between each oscillator.**

## I. Introduction

Coupled oscillatory systems are good model to express high dimensional nonlinear phenomena in the natural science fields. There are many studies on synchronization phenomena of coupled oscillator systems in many research fields [1]-[7]. So, many kind of coupled systems have been investigated to analyze the mechanism of the nonlinear phenomena. We can see that coupled oscillatory systems produce interesting phase wave patterns, including wave propagation, clustering and complex patterns. However, synchronization phenomena of the oscillators have not been analyzed enough yet. Therefore, we have to investigate the case of more complicated synchronization phenomena to elucidate high dimensional nonlinear phenomena.

In our previous works, we have investigated synchronization phenomena in three coupled van der Pol oscillators as a ring topology [8]. In this circuit system, the number of coupled oscillators is an odd number. Then coupled oscillators can not synchronize with in/anti phase states. In other words, three-phase synchronization (phase shift: 120°) is obtained by effect of frustration. However, three-phase synchronization is always observed stably in this system.

In our study, we suppose that we can survey several kinds of interesting synchronization phenomena in coupled oscillatory system which has stronger frustrations. We research four coupled van der Pol oscillators in the form of the regular tetrahedron as shown in Fig. 1. In addition, we calculate the phase differences of each oscillator in four coupled van der Pol oscillators. By changing the parameters of the nonlinearity and the coupling strength, we observe how sojourn time in in-phase or anti-phase changes.

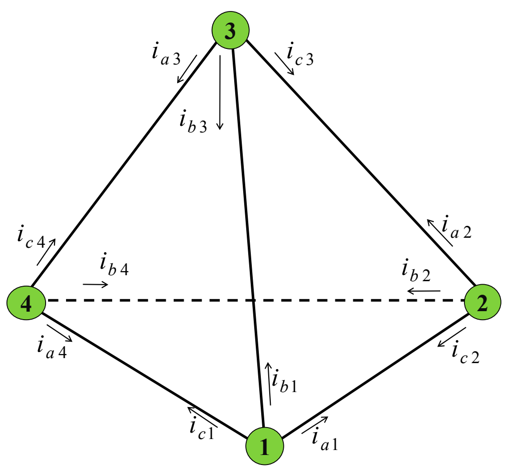


Fig. 1. Conceptual circuit model for tetrahedron form.

## II. Circuit Model

The circuit model of four coupled van der Pol oscillators in the form of the regular tetrahedron is shown in Fig. 2. Van der Pol oscillator is constituted an inductor, a negative resistor and a capacitor. The adjacent oscillators are coupled by resistor via inductors.

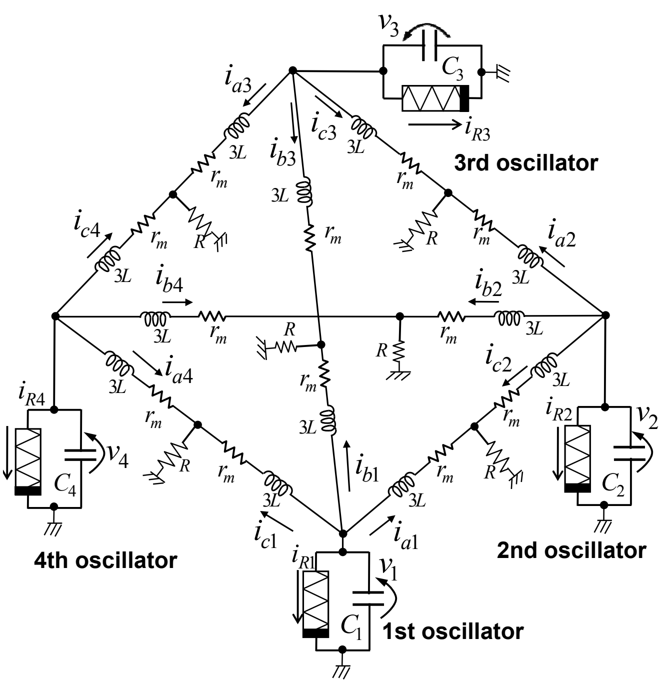


Fig. 2. Four coupled van der Pol oscillators.

In this figure, we assume that the electric currents flowing from each top are $i_{ak}$, $i_{bk}$ and $i_{ck}$, the voltages of the capacitor of each oscillator are $v_k$ and the electric current flowing from the negative resistance $i_{Rk}$. When we couple four oscillators, tiny resistor $r_m$ is introduced to consider internal resistance. In addition, we connect tiny resistor to avoid *L*-loop. If *L*-loop occurs in four coupled oscillators, the computer simulations can not carry out normally.

In the computer simulations, we assume that the $v_k - i_{Rk}$ characteristics of nonlinear resistor in each oscillator is given by the following third order polynomial equation.

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3 \qquad (g_1, g_3 > 0), \; (k = 1, 2, 3, 4). \tag{1}$$

The normalized circuit equations are expressed as:

$$\begin{cases} \dfrac{dx_k}{d\tau} = \varepsilon(1 - {x_k}^2)x_k - (y_{ak} + y_{bk} + y_{ck}) \\ \dfrac{dy_{ak}}{d\tau} = \dfrac{1}{3}\{x_k - \eta y_{ak} - \gamma(y_{ak} + y_{c(k+1)})\} \\ \dfrac{dy_{bk}}{d\tau} = \dfrac{1}{3}\{x_k - \eta y_{bk} - \gamma(y_{bk} + y_{b(k+2)})\} \\ \dfrac{dy_{ck}}{d\tau} = \dfrac{1}{3}\{x_k - \eta y_{ck} - \gamma(y_{ck} + y_{a(k+3)})\}. \end{cases} \tag{2}$$

We use the following normalizations:

$$t = \sqrt{LC}\tau \,, v_k = \sqrt{\frac{g_1}{g_3}}\, x_k \,, i_{ak} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{ak},$$

$$i_{bk} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{bk} \,, i_{ck} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{ck},$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}} \,, \gamma = R\sqrt{\frac{C}{L}} \,, \eta = r_m\sqrt{\frac{C}{L}},$$

(k=1,2,3,4),

where $\varepsilon$ is the degree of nonlinearity, $\gamma$ is the coupling strength, and $\eta$ indicates the resistive component. In the computer simulations, we investigate the phase differences between adjacent oscillators.

## III. Synchronization Phenomena

We carry out computer simulations for the four coupled van der Pol oscillators in the regular tetrahedron form circuit. Figure 3 shows the time wave form of four coupled van der Pol oscillators. The horizontal axis shows the computer simulation time and the vertical axis shows the time wave form of the voltage. In this case, we input the parameters $\varepsilon$ = 0.50, $\eta$ = 0.02 and $\gamma$ = 0.15. Figure 4 shows the phase plane between the adjacent oscillators at same time as Fig. 3.

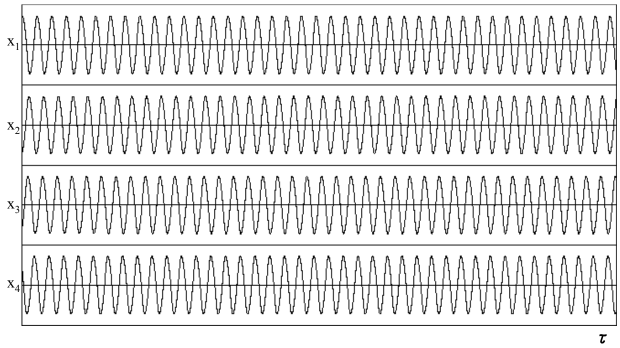

Fig. 3. The time wave form of the voltage.

In addition, we observe interesting synchronization phenomena that the phase differences between oscillators change periodically from 0° to 180° or the reverse. The synchronization states stay at the anti-phase whose sojourn time is longer than that of the in-phase.

Figure 5 shows one example of the changing phase differences for the parameters $\varepsilon$ = 0.50, $\eta$ = 0.02 and $\gamma$ = 0.15 (Fig. 5(a)) and $\gamma$ = 0.40 (Fig. 5(b)). This figure shows the change of the phase difference for time. The horizontal axis shows the computer simulation time and the vertical axis shows the phase difference of each oscillator. In Fig. 5, "1-2" indicates a phase difference between the first oscillator and the second oscillator.

In this figure, we confirm that the phase differences of all oscillators change periodically. In addition, we can see that sojourn time of the in-phase is instant whereas the anti-phase is relatively long. To increase $\gamma$, periods of the shifting the phase

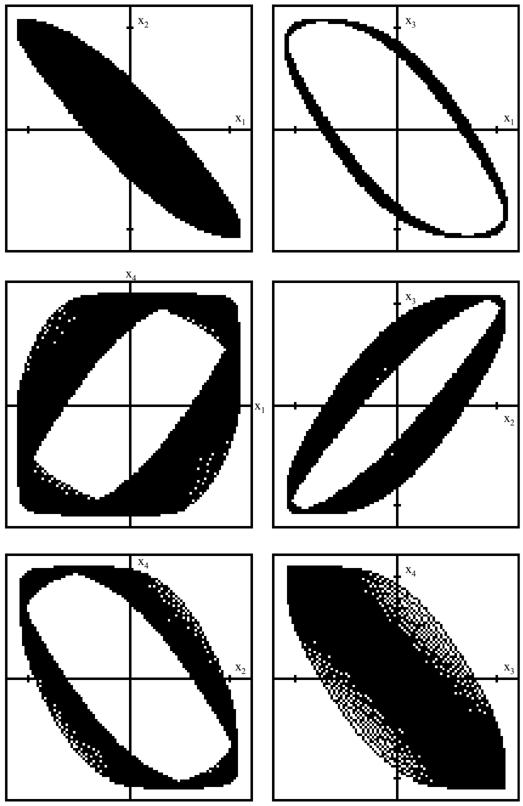
Fig. 4. Lissajous figures.

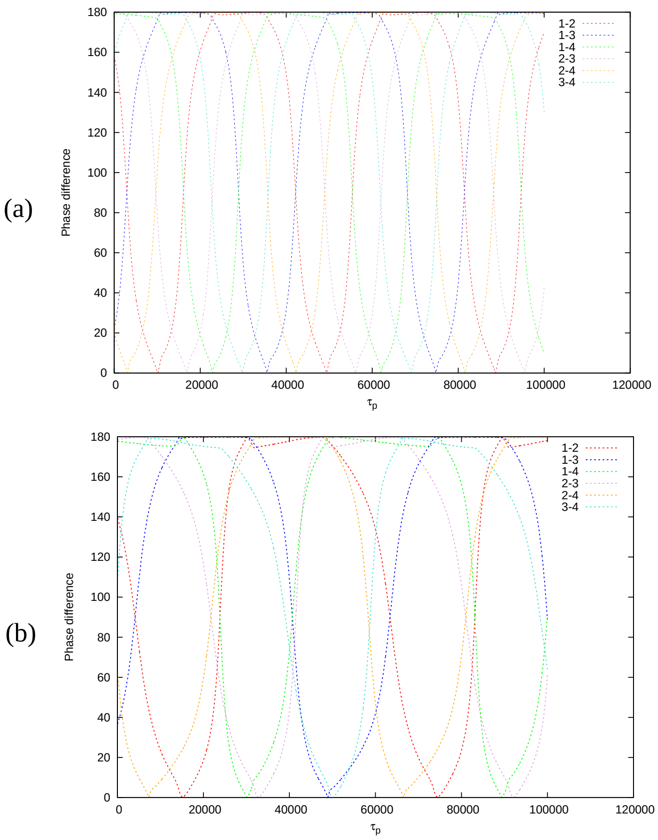


Fig. 5. The phase difference between oscillators for $\varepsilon = 0.50$ and $\eta = 0.02$. (a)$\gamma = 0.15$. (b)$\gamma = 0.40$.

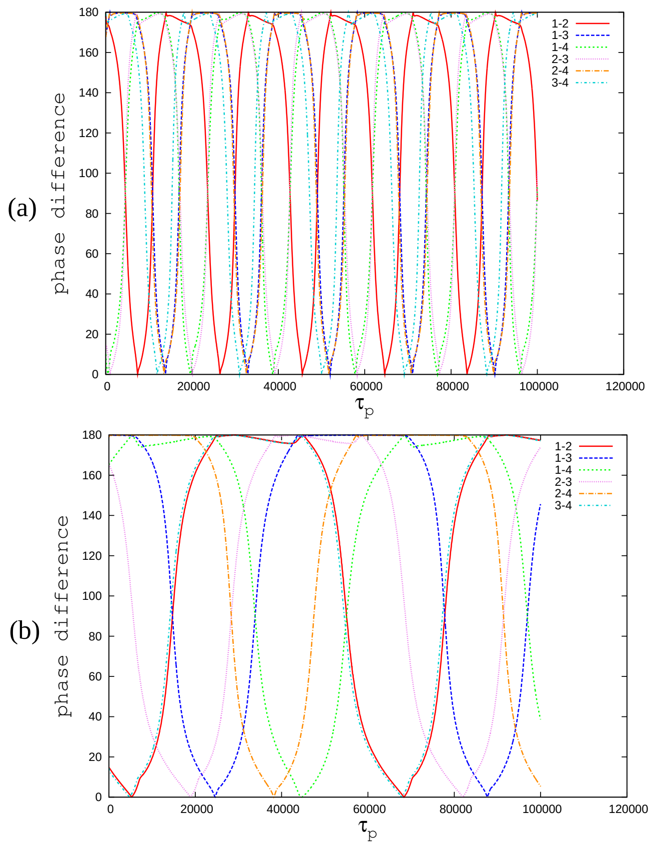


Fig. 6. The phase difference between oscillators for $\gamma = 0.40$ and $\eta = 0.02$. (a)$\varepsilon = 0.20$. (b)$\varepsilon = 0.40$.

differences become longer. In the case of Fig. 5(b), the interval of the in-phase becomes long in comparison with Fig. 5(a).

In next simulation, we change the parameters. We fix $\gamma$ and carry out simulation whereas we fixed $\varepsilon$ in former computer simulation. Figure 6 shows the simulation results for the parameters $\gamma = 0.40$, $\eta = 0.02$ and $\varepsilon = 0.20$ (Fig. 6(a)) and $\varepsilon = 0.40$ (Fig. 6(b)). In comparison with Fig 6(a) and Fig 6(b), we can understand that there is time lag which switch of in-phase or anti-phase. In particular, the switching time of anti-phase becomes long to increase $\varepsilon$.

## IV. Conclusions

In this study, we investigated synchronization phenomena observed in the four coupled oscillators in form of a regular tetrahedron. We observed synchronisation phenomena that the phase differences changed periodically. Moreover, when the coupled strength was small, changes of phase differences became sharp. The anti-phase state continued for a long time with increasing $\gamma$.

## Acknowledgment

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