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Nonlinear Spring Model of Self-Organizing Map and its Chaotic Behavior

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Abstract The Self-Organizing Map (SOM) is an unsupervised neural network introduced by Kohonen and is a model simplifying self-organization process of the brain. However, SOM is still far away from the realization of the brain mechanism. In order to realize more powerful and more flexible mechanism, it is important to propose new models of the brain mechanism and to investigate their behaviors. In this study, as the first step to realize a new nonlinear spring model of SOM, we propose a simple one dimensional 2-neuron model connected by a nonlinear spring. We investigate its behavior under a simple assumption where input vectors are given to the model periodically. Furthermore, in order to investigate the behavior of the nonlinear spring model of SOM, we calculate one-parameter bifurcation diagram and the largest Lyapunov exponent of the proposed model. Computer simulated results show that the neurons oscillate chaotically.

Key words Self-Organizing Maps (SOM), Clustering, Data Mining

1. Introduction

The Self-Organizing Map (SOM) is a subtype of artificial neural networks. It is trained using unsupervised learning to produce low dimensional representation of the training samples while preserving the topological properties of the input space. SOM is introduced by Kohonen in 1982 [1] and is a model simplifying self-organization process of the brain.

However, SOM is still far away from the realization of the brain mechanism. In order to realize more powerful and more flexible mechanism, it is important to propose new models of the brain mechanism and to investigate their behaviors.

In this study, as the first step to realize a new nonlinear spring model of SOM, we propose a simple one dimensional 2-neuron model connected by a nonlinear spring. We investigate its behavior under a simple assumption where input vectors are given to the model periodically. Furthermore, in order to investigate the behavior of the nonlinear spring model of SOM, we calculate one-parameter bifurcation diagram and the largest Lyapunov exponent of the proposed model. In Section II, we explain the learning algorithm of SOM. In Section III, we explain proposed nonlinear spring model of SOM in detail. The behaviors of the proposed model are investigated by calculating the projection of attractors and Poincaré map in Section IV. Furthermore, in order to investigate chaotic behavior in detail, one-parameter bifurcation diagram and the largest Lyapunov exponent are calculated. Computer simulated results show that the neurons oscillate chaotically.

2. Self-Organizing Map (SOM)

We explain the learning algorithm of the Self-Organizing Map (SOM). SOM consists of m neurons located at a regular low-dimensional grid, usually a 2-D grid. The basic SOM algorithm is iterative. Each neuron i has a d-dimensional weight vector $\boldsymbol{w}_i = (w_{i1}, w_{i2}, \cdots, w_{id})$ $(i = 1, 2, \cdots, m)$. The initial values of all the weight vectors are given over the input space at random. The range of the elements of d-dimensional input data $\boldsymbol{x}_j = (x_{j1}, x_{j2}, \cdots, x_{jd})$ $(j = 1, 2, \cdots, N)$ are assumed to be from 0 to 1.

(SOM1) An input vector $\boldsymbol{x}_j$ is inputted to all the neurons at the same time in parallel.
(SOM2) Distances between $\boldsymbol{x}_j$ and all the weight vectors are calculated. The winner, denoted by c, is the neuron with the weight vector closest to the input vector $\boldsymbol{x}_j$;

$$c = \arg\min_i \{\|\boldsymbol{w}_i - \boldsymbol{x}_j\|\}, \tag{1}$$

where $\|\cdot\|$ is the distance measure, Euclidean distance.
(SOM3) The weight vectors of the neurons are updated as;

$$\boldsymbol{w}_i(t+1) = \boldsymbol{w}_i(t) + h_{c,i}(t)(\boldsymbol{x}_j - \boldsymbol{w}_i(t)), \tag{2}$$

where t is the learning step. $h_{c,i}(t)$ is called the neighborhood function and is described as a Gaussian function;

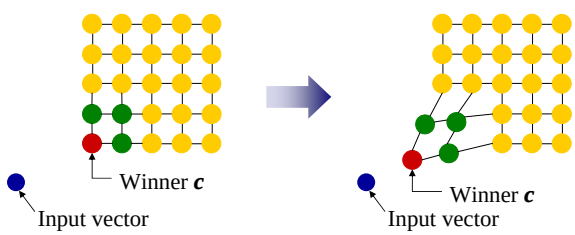


Fig. 1 Learning process of Self-Organizing Map.

$$h_{c,i}(t) = \alpha(t) \exp\left(-\frac{\|\boldsymbol{r}_i - \boldsymbol{r}_c\|^2}{2\sigma^2(t)}\right), \tag{3}$$

where $\|\boldsymbol{r}_i - \boldsymbol{r}_c\|$ is the distance between map nodes c and i on the map grid, $\alpha(t)$ is the learning rate, and $\sigma(t)$ corresponds to the width of the neighborhood function. Both $\alpha(t)$ and $\sigma(t)$ decrease with time.

(SOM4) The steps from (SOM1) to (SOM3) are repeated for all the input data.

3. Nonlinear Spring Model of SOM

We consider a simple one dimensional 2-neuron model as the first step to realize the nonlinear spring model of SOM. The model is shown in Fig. 2. The two neurons are assumed to have the same mass m and to be connected by a nonlinear spring with the natural length l whose restoring force F against the variation x is represented by

$$F = -bx^3 \tag{4}$$

where b denotes the stiffness of the spring.

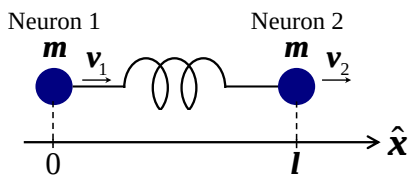


Fig. 2 Two-neuron nonlinear spring model of SOM.

Without loss of generality, we fix the position of the Neuron 1 as the origin of the x-coordinate. The position of the Neuron 2 ($\hat{x}$) and the velocities of the neurons (v_1 and v_2) are chosen as the state variables. The motion equation of the model can be described as

$$\begin{cases} \dfrac{d\hat{x}}{dt} &= v_2 \\ m\dfrac{dv_1}{dt} &= -av_1 + b(\hat{x} - l)^3 \\ m\dfrac{dv_2}{dt} &= -av_2 - b(\hat{x} - l)^3 \end{cases} \tag{5}$$

where a is the friction parameter. By changing the variables and parameters;

$$\begin{aligned} &\hat{x} - l = x, \quad v_1 = \sqrt{\frac{b}{m}}\, y_1, \quad v_2 = \sqrt{\frac{b}{m}}\, y_2, \\ &t = \sqrt{\frac{m}{b}}\, \tau, \quad k = \frac{a}{\sqrt{bm}}, \end{aligned} \tag{6}$$

the normalized equations are given as

$$\begin{cases} \dfrac{dx}{d\tau} &= y_2 \\ \dfrac{dy_1}{d\tau} &= -ky_1 + x^3 \\ \dfrac{dy_2}{d\tau} &= -ky_2 - x^3. \end{cases} \tag{7}$$

Next, we model the learning process of the SOM by the external force by input vectors. When an input vector is given to the 2-neuron model as Fig. 3, the winner which is the neuron nearer to the input vector is attracted to the input with the following force;

$$\hat{f}(t) = \hat{B} \sin\sqrt{\frac{b}{m}}\, t \qquad \left(0 \leqq t < \sqrt{\frac{m}{b}}\, \pi\right) \tag{8}$$

where $t = 0$ is the time when the input vector is given. The shape of this function is shown in Fig. 4. Note that the other neuron does not receive a direct effect from the input vector.

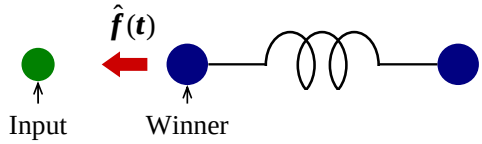


Fig. 3 Input vector and winner.

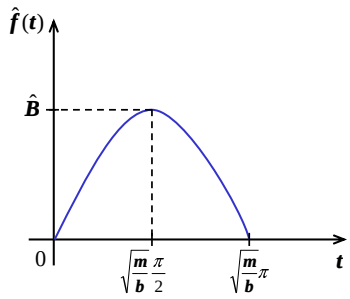


Fig. 4 Force by external input vector.

In this study, in order to investigate the simplest learning process of the 2-neuron model, we concentrate on the case that input vectors are given to the right-hand side and the left-hand side of the model alternately with the fixed frequency $\sqrt{b/m}/(2\pi)$. In this case, the motion equation is modified as

$$\begin{cases} \dfrac{dx}{d\tau} &= y_2 \\ \dfrac{dy_1}{d\tau} &= -ky_1 + x^3 - f(\tau) \\ \dfrac{dy_2}{d\tau} &= -ky_2 - x^3 + f(\tau - \pi), \end{cases} \tag{9}$$

where

$$f(\tau) = \frac{B}{2}(\sin\tau + |\sin\tau|), \tag{10}$$

and

$$B = \frac{\hat{B}}{b}. \tag{11}$$

The shape of $f(\tau)$ is shown in Fig. 5.

4. Computer Simulation Results

In this section, we show some computer calculation results obtained by using Runge-Kutta method with time step $\delta t = 2\pi/500$ for Eq. (9).

4.1 Attractors and Poincaré maps

The projections of attractors onto x–y_2 plane are shown in Fig. 6(a). By changing parameter B, we can see that four-periodic orbit (1), one-periodic orbit (3) and two-periodic orbit (4). Furthermore, we can confirm that the orbits of the attractors (2), (5) and (6) look chaos.

In order to investigate the chaotic behavior of the model in detail, we define the Poincaré section as $\tau = 2n\pi$ and plot the discrete data on the Poincaré section onto x–y_2 plane. The Poincaré maps are shown in Fig. 6(b). Four-periodic orbit (1) bifurcates to chaos which is estimated from the shape of the Poincaré map. Periodic window (3) and (4) can be observed in the chaos region. As B increases, the chaos (5) grows to more complex chaos (6). We can see that the Poincaré map (5) and (6) are folded and have the shape like strange attractors [2].

4.2 Bifurcation Diagram and Lyapunov Exponent

In order to calculate the largest Lyapunov exponents of the attractors, we derive the variational equations of Eq. (9) as follows;

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial x}{\partial x_0} = \dfrac{\partial y_2}{\partial x_0} \\ \dfrac{d}{d\tau}\dfrac{\partial y_1}{\partial x_0} = -k\dfrac{\partial y_1}{\partial x_0} + 3x^2\dfrac{\partial x}{\partial x_0} \\ \dfrac{d}{d\tau}\dfrac{\partial y_2}{\partial x_0} = -k\dfrac{\partial y_2}{\partial x_0} - 3x^2\dfrac{\partial x}{\partial x_0} \\ \dfrac{d}{d\tau}\dfrac{\partial x}{\partial y_{1_0}} = \dfrac{\partial y_2}{\partial y_{1_0}} \\ \dfrac{d}{d\tau}\dfrac{\partial y_1}{\partial y_{1_0}} = -k\dfrac{\partial y_1}{\partial y_{1_0}} + 3x^2\dfrac{\partial x}{\partial y_{1_0}} \\ \dfrac{d}{d\tau}\dfrac{\partial y_2}{\partial y_{1_0}} = -k\dfrac{\partial y_2}{\partial y_{1_0}} - 3x^2\dfrac{\partial x}{\partial y_{1_0}} \\ \dfrac{d}{d\tau}\dfrac{\partial x}{\partial y_{2_0}} = \dfrac{\partial y_2}{\partial y_{2_0}} \\ \dfrac{d}{d\tau}\dfrac{\partial y_1}{\partial y_{2_0}} = -k\dfrac{\partial y_1}{\partial y_{2_0}} + 3x^2\dfrac{\partial x}{\partial y_{2_0}} \\ \dfrac{d}{d\tau}\dfrac{\partial y_2}{\partial y_{2_0}} = -k\dfrac{\partial y_2}{\partial y_{2_0}} - 3x^2\dfrac{\partial x}{\partial y_{2_0}}. \end{cases} \tag{12}$$

The Jacobian Matrix **DT** are obtained by integrating Eq. (12) numerically [3];

$$\mathbf{DT} = \begin{bmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_{1_0}} & \dfrac{\partial x}{\partial y_{2_0}} \\ \dfrac{\partial y_1}{\partial x_0} & \dfrac{\partial y_1}{\partial y_{1_0}} & \dfrac{\partial y_1}{\partial y_{2_0}} \\ \dfrac{\partial y_2}{\partial x_0} & \dfrac{\partial y_2}{\partial y_{1_0}} & \dfrac{\partial y_2}{\partial y_{2_0}} \end{bmatrix}. \tag{13}$$

By using the Jacobian Matrix **DT**, we can calculate the

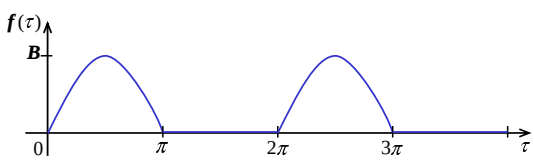


Fig. 5 Force by periodic external input vector.

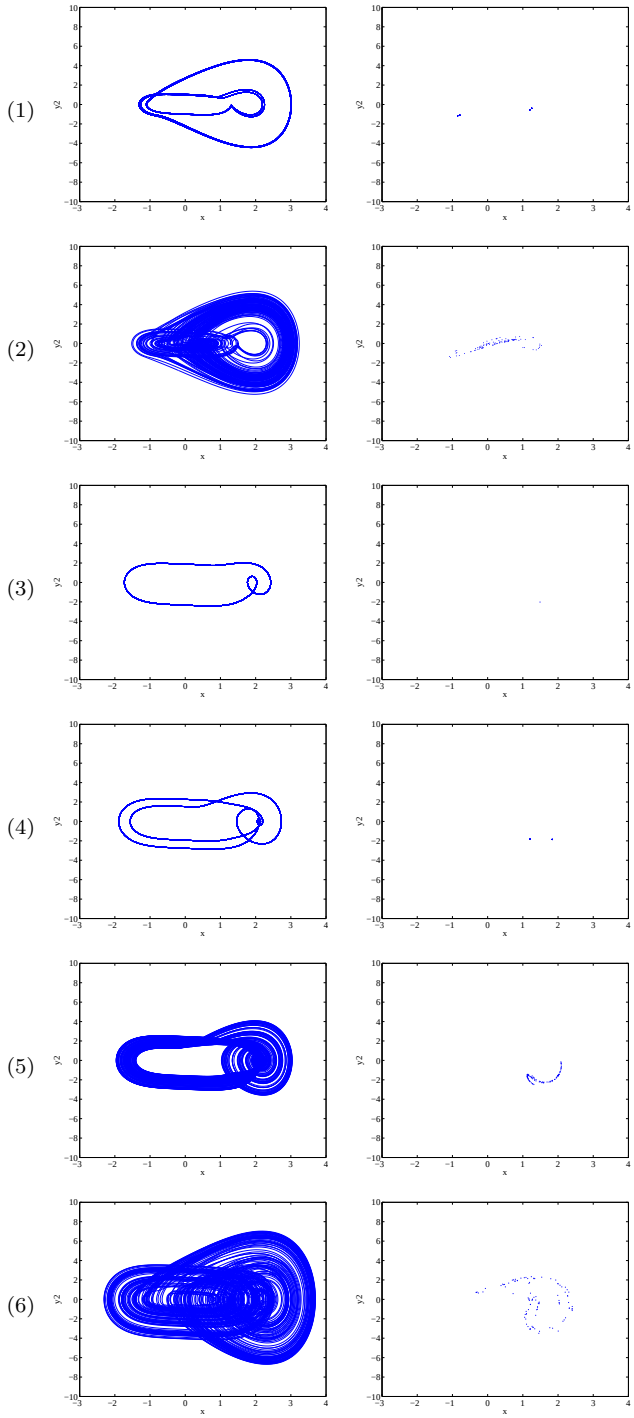


Fig. 6 Projection of attractors onto x–y_2 plane and their Poincaré maps. Fixed parameter $k = 0.15$. (a) Attractors. (b) Poincaré maps. (1) $B = 6.75$. (2) $B = 8$. (3) $B = 9.5$. (4) $B = 10$. (5) $B = 10.4$. (6) $B = 12$.

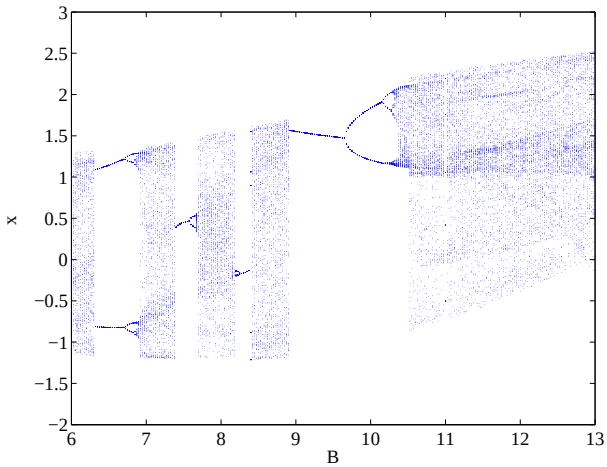
largest Lyapunov exponent;

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \log |\mathbf{DT}(j) \cdot \boldsymbol{e}(j)|, \tag{14}$$

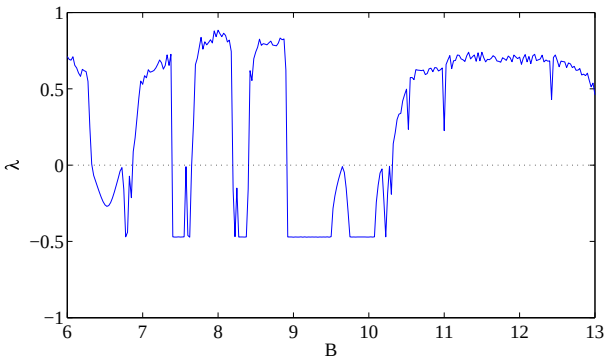
where $\boldsymbol{e}(j)$ is a normalized basis.

One-parameter bifurcation diagram of x and the largest Lyapunov exponents of the attractors, which are calculated using the algorithm of Shimada-Nagashima [4], are shown in

Figs. 7(a) and (b), respectively. The control parameter is B and another parameter k is fixed as 0.15. As shown in Fig. 7(b), the Lyapunov exponent takes positive values for a wide range of B. Therefore, the nonlinear spring model can be said to generate chaos, namely, the neurons oscillate chaotically. We observe the bifurcation in detail. For $6.3 < B < 6.92$, a window corresponding to the periodic orbit in Fig. 6(1) is observed. For $6.92 < B < 7.4$, the largest Lyapunov exponent takes positive values, namely, the attractor in Fig. 6(2) is chaos. For $8.9 < B < 10.5$, the large window corresponding to the period-doubling bifurcation in Figs. 6(3) and (4) is observed. For $B < 10.5$, chaotic attractors can be observed for almost parameter values. We have also confirmed that a lot of small periodic windows are embedded in the chaotic region.



(a)



(b)

Fig. 7 Bifurcation diagram and Lyapunov exponent of the nonlinear spring model. (a) One-parameter bifurcation diagram. (b) Largest Lyapunov exponent. $k = 0.15$.

5. Conclusions

In this study, as the first step to realize a new nonlinear spring model of SOM, we have proposed a simple one dimensional 2-neuron model connected by a nonlinear spring. We have investigated its behavior under a simple assumption where input vectors are given to the model periodically. Furthermore, we have investigate the behavior of the nonlinear spring model of SOM by calculating the projection of attractor and Poincaré map. In order to investigate chaotic behavior in detail, one-parameter bifurcation diagram and the largest Lyapunov exponent were calculated. We have investigated the related bifurcation phenomena and have confirmed that the neurons oscillate chaotically.

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