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SYNCHRONIZATION PHENOMENA IN VAN DER POL OSCILLATORS COUPLED BY A TIME-VARYING RESISTOR

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In this study, synchronization phenomena observed in van der Pol oscillators coupled by a time-varying resistor are investigated. We realize the time-varying resistor by switching a positive and a negative resistor periodically. By carrying out circuit experiments and computer calculations, interesting synchronization phenomena can be confirmed to be generated in this system. Namely, the synchronization states change according to the switching frequency of the time-varying resistor.

Keywords: Coupled oscillators; synchronization; time-varying resistor.

1. Introduction

Synchronization phenomena in complex systems are very interesting to describe various higher-dimensional nonlinear phenomena in the field of natural science. Studies on synchronization phenomena of coupled oscillators are extensively carried out in various fields, such as in physics [Bonilla *et al.*, 1998; Sherratt, 1998; Abramson *et al.*, 2002; Belykh, 2005], biology [Gray, 1994; Cosp *et al.*, 2004] and engineering [Suezaki & Mori, 1965; Kimura & Mano, 1965; Endo & Mori, 1976a, 1976b, 1978; Datardina & Linkens, 1978; Nishio & Mori, 1992; Yamauchi *et al.*, 1999; Lin *et al.*, 2005]. Furthermore many researchers suggest that synchronization phenomena of coupled oscillators have some relations to information processing in the brain. We consider that it is very important to investigate the synchronization phenomena of coupled oscillators to realize a brain computer for future engineering applications.

On the other hand, there are some systems whose dissipation factors vary with time, for example, under the time-variation of the ambient temperature, an object moving in a space with some friction and a circuit with a resistor whose temperature coefficient is sensitive such as thermistor. However, oscillators coupled by a time-varying resistor have been barely studied.

In this study, synchronization phenomena observed in van der Pol oscillators coupled by a time-varying resistor are investigated. We realize the time-varying resistor by switching a positive and a negative resistor periodically. By carrying out circuit experiments and computer calculations, interesting synchronization phenomena can be confirmed to be generated in this system. Further, we investigate the influence of the duty ratio of the switching, namely we vary the ratio of time intervals connecting to the positive and the negative resistors. Various interesting features of the coupled system can be clarified.

2. Circuit Model

Figure 1 shows the circuit model. In this circuit, two identical van der Pol oscillators are coupled by

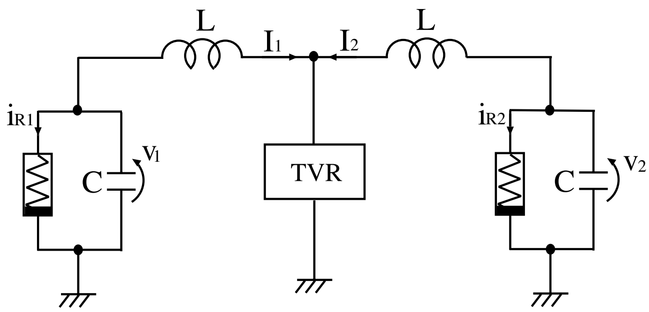


Fig. 1. Circuit model (TVR is a time-varying resistor).

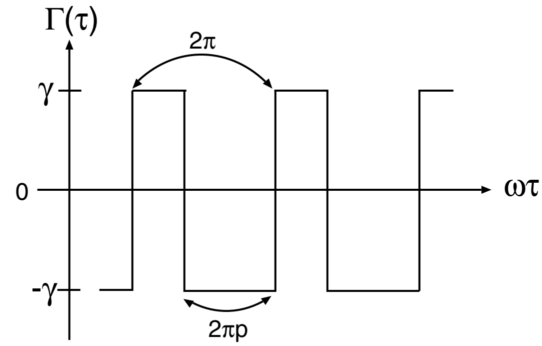


Fig. 3. Normalized characteristics of the TVR.

a Time-Varying Resistor (TVR). Synchronization phenomena for the case that the coupling resistor is a simple time-invariant resistor have been investigated in [Suezaki & Mori, 1965; Kimura & Mano, 1965]. Namely, the in-phase synchronization is stable for a positive coupling resistor, while the anti-phase synchronization is stable for a negative coupling resistor. In this study, we consider the case that the coupling resistance $R(t)$ varies with time. The characteristics of the TVR are shown in Fig. 2. In this study, we consider the case that the function representing the variation of the VTR is a square wave with the angular frequency ω_t and the duty ratio p.

Firstly, the $v_k - i_{Rk}$ characteristics of the nonlinear resistor are defined as follows,

$$i_{Rk} = -g_1 v_k + g_3 v_k^3. \tag{1}$$

By means of the following change of variables,

$$v_k = \sqrt{\frac{g_1}{g_3}} x_k, \quad i_k = \sqrt{\frac{g_1}{3g_3}} \sqrt{\frac{C}{L}} y_k, \quad t = \sqrt{LC}\tau,$$

$$\varepsilon = g_1 \sqrt{\frac{L}{C}}, \quad \gamma = r\sqrt{\frac{C}{L}}, \quad \omega = \frac{1}{\sqrt{LC}}\omega_t,$$

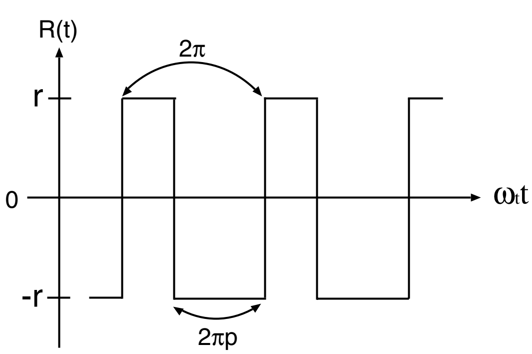


Fig. 2. Characteristics of the TVR.

the normalized circuit equations are given by

$$\begin{cases} \dfrac{dx_k}{d\tau} = \epsilon(1 - x_k^2) - y_k \\ \dfrac{dy_k}{d\tau} = x_k - \Gamma(\tau)\displaystyle\sum_{j=1}^{2} y_j \end{cases} \quad (k = 1, 2) \tag{2}$$

where $\Gamma(\tau)$ corresponds to the normalized characteristics of the time-varying resistor and is shown in Fig. 3.

3. Synchronization Phenomena

3.1. *In-phase and anti-phase synchronization*

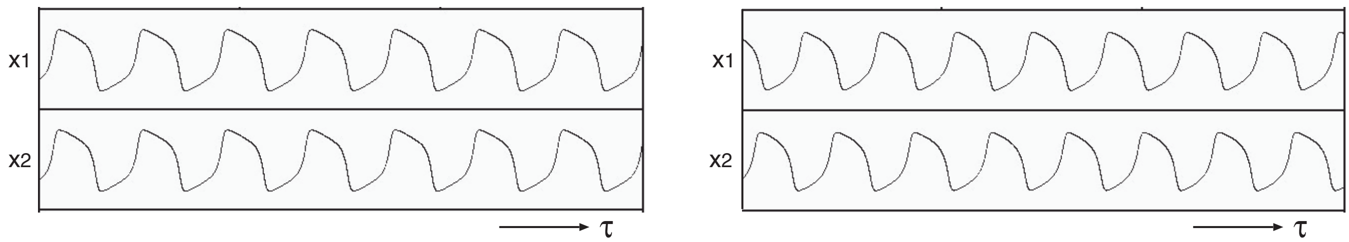
The two coupled oscillators can be synchronized in in-phase and anti-phase as shown in Fig. 4. These two synchronization states can be obtained starting at different initial conditions.

We have confirmed the two synchronization states in-phase and anti-phase in circuit experiments, as shown in Fig. 5. In the circuit experiments, the TVR is realized by using an analog switch as shown in Fig. 6 [Nishio & Mori, 1993].

3.2. *Synchronization in dependence on ω*

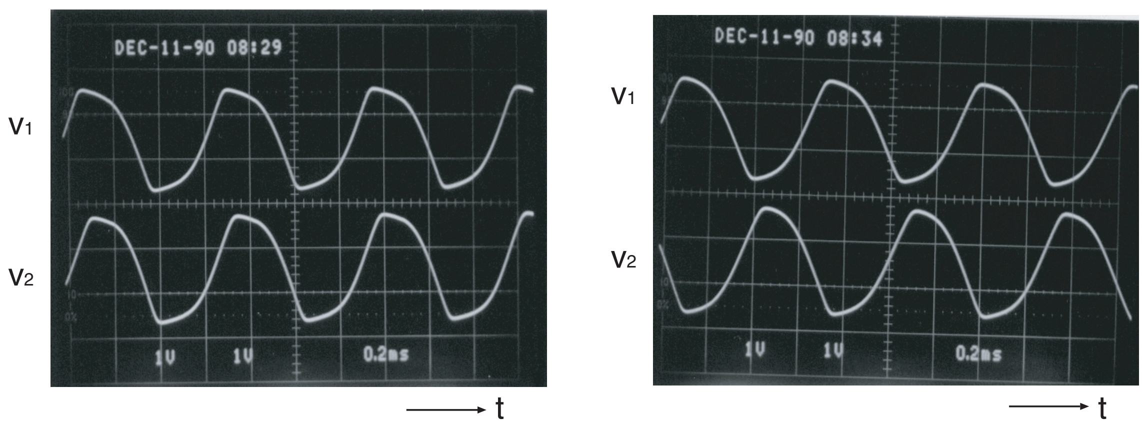
We investigate how the synchronization state depends on ω. The computer simulation is shown in Fig. 7. The horizontal axis is ω and the vertical axis is the phase difference between the two coupled oscillators. We set the parameters of the circuits as $\varepsilon = 2.0$, $\gamma = 0.1$ and $p = 0.5$.

From Fig. 7, we can confirm the coexistence of in-phase and anti-phase synchronizations for $1.44 < \omega < 1.58$. We also confirmed that the range of the coexistence region shifts toward smaller ω when ε increases.



(a) In-phase. (b) Anti-phase.

Fig. 4. Time waveform of two synchronization states (computer simulation). $\varepsilon = 2.0, \gamma = 0.1, \omega = 1.5, p = 0.5$. (a) In-phase synchronization (the initial states: $x_1 = 1.0$, $y_1 = 1.0$, $x_2 = 1.1$, $y_2 = 1.1$). (b) Anti-phase synchronization (the initial states: $x_1 = 1.0$, $y_1 = 1.0$, $x_2 = -1.1$, $y_2 = -1.1$).



(a) In-phase. (b) Anti-phase.

Fig. 5. Time waveform of two synchronization states (circuit experimental results). $L = 10\,\text{mH}$, $C = 33\,\text{nF}$, $r = 152\,\Omega$. (a) In-phase synchronization, (b) Anti-phase synchronization.

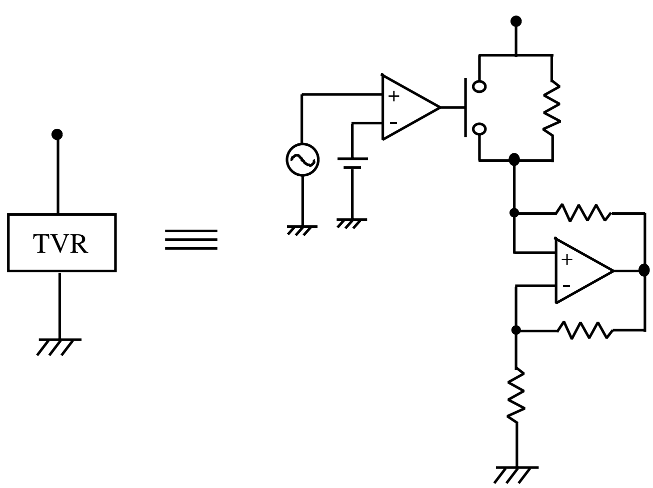


Fig. 6. Circuit realization of the TVR [Nishio & Mori, 1993].

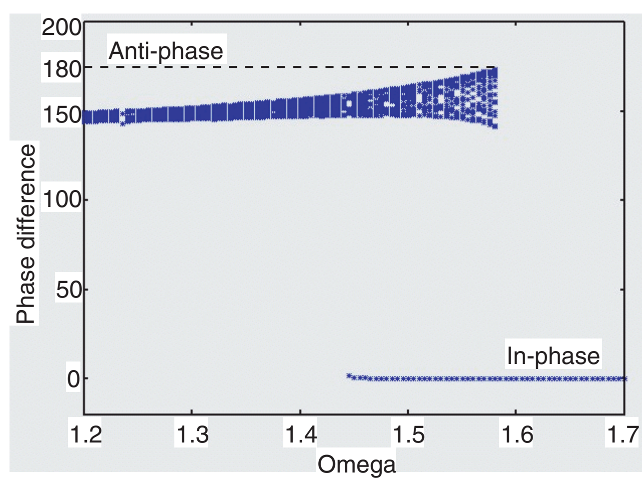


Fig. 7. Synchronization in dependence on ω ($\epsilon = 2.0$, $\gamma = 0.1$, $p = 0.5$).

It is also interesting to see that the anti-phase synchronization is not completely anti-phase. The solution oscillates around 150° and oscillation amplitude becomes larger as ω increases. This means that another anti-phase synchronization mode coexists around 210° because of the symmetry of the coupled circuits. Anti-phase synchronization becomes unstable around $\omega = 1.58$,

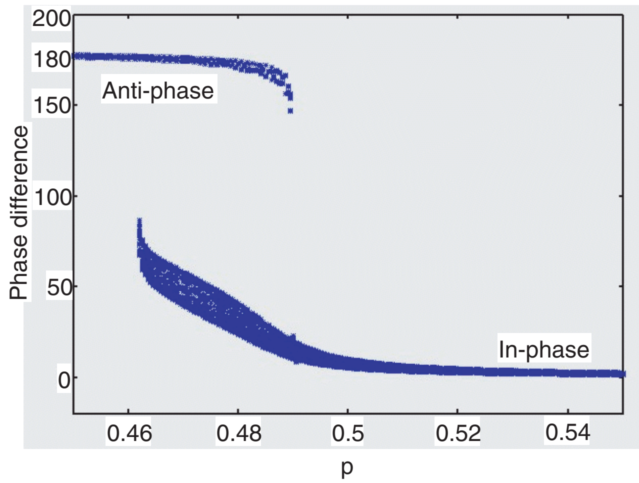


Fig. 8. Synchronization in dependence on p ($\epsilon = 2.0$, $\gamma = 0.1$, $\omega = 2.0$).

where two anti-phase synchronizations seem to collide with each other. We consider that the detailed study on this bifurcation would help to clarify the mechanism of the coexistence of synchronization modes in this circuits.

3.3. *Synchronization in dependence on p*

Next, we investigate the influence of the duty ratio of the switching, namely we vary the ratio of time intervals connecting to the positive and the negative resistors. Figure 8 shows the phase difference between both oscillators in dependence on the duty ratio p obtained from Eq. (2). We set the parameters as $\epsilon = 2.0$, $\gamma = 0.1$ and $\omega = 2.0$.

From Fig. 8, we can see the hysteresis of the synchronization phenomena. First, for p smaller than 0.462, the two coupled oscillators are synchronized only in anti-phase. Second, when the p is between 0.462 and 0.490, the coexistence of the in-phase and the anti-phase synchronizations can be observed. Finally, for p larger than 0.490, only the in-phase synchronization can be confirmed.

It should be noted that the in-phase synchronization is not completely in-phase in the coexistence region. The phase difference becomes larger as p decreases.

The width of the coexistence region depends on the values of the parameter ϵ, which is responsible for the nonlinearity of the oscillators. We confirmed that the width of the coexistence region becomes narrower, when ϵ increases.

4. Conclusions

In this study, we investigated synchronization phenomena of two van der Pol oscillators coupled by a time-varying resistor in dependence on ω and p. We realized the time-varying resistor by switching a positive and a negative resistor periodically. By carrying out circuit experiments and computer calculations, we confirmed that the range of the coexistence region shifts toward smaller ω, and the width of the coexistence region becomes narrower, when ϵ increases. A detailed study of the found bifurcations will be the subject of a forthcoming paper.

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