PAPER

Wave Propagation Phenomena of Phase States in Oscillators Coupled by Inductors as a Ladder

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SUMMARY In this study, wave propagation phenomena of phase states are observed at van der Pol oscillators coupled by inductors as a ladder. For the case of 17 oscillators, interesting wave propagation phenomena of phase states are found. By using the relationship between phase states and oscillation frequencies, the mechanisms of the propagation and the reflection of wave are explained. Circuit experimental results agree well with computer calculated results qualitatively.

key words: *coupled oscillators, wave propagation phenomena, phase-difference, oscillation frequency, wave propagation speed*

1. Introduction

A lot of studies on synchronization phenomena of coupled oscillators have been carried out up to now. Endo et al. have reported a details of theoretical analysis and circuit experiments about some coupled oscillators as a ladder, ring and two-dimensional array [1]–[3]. Recently, wave propagation phenomenon observed from coupled chaotic circuits is also reported [4], [5]. However, such studies treat only transient states for a given set of initial conditions and there seems to be very few studies on continuously existing wave propagation phenomenon observed simple coupled oscillators circuits.

In this study, we investigate wave propagation phenomena observed from van der Pol oscillators coupled by inductors as a ladder. By computer calculations for the case of 17 oscillators, we can find various interesting wave propagation phenomena of phase states. By using the relationship between phase states and oscillation frequencies, we can explain why does the wave propagation and why does the wave reflection. Further, for the case of 5 oscillators, we carry out both computer calculations and circuits experiments. Circuit experimental results agree well with computer calculated results qualitatively.

2. Circuit Model

Circuit model is shown in Fig. 1. N van der Pol oscillators are coupled by coupling inductors L_0. We carried out computer calculations for the cases of $N = 5 \cdots 17$ and circuit experiments for the case of $N = 5$. In the computer calculations, we assume the $v-i$ characteristics of nonlinear negative resistors in the circuit by the following functions.

$$i_r(v_k) = -g_1 v_k + g_3 v_k^3 \quad (g_1, g_3 > 0) \tag{1}$$

The circuit equations governing the circuit in Fig. 1 are expressed as

[First Oscillator]

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= -x_1 + \alpha(x_2 - x_1) + \varepsilon\left(y_1 - \frac{1}{3}y_1^3\right) \end{aligned} \tag{2}$$

[Middle Oscillators]

$$\begin{aligned} \dot{x}_k &= y_k \\ \dot{y}_k &= -x_k + \alpha(x_{k+1} - 2x_k + x_{k-1}) + \varepsilon\left(y_k - \frac{1}{3}y_k^3\right) \\ &\qquad (k = 2, 3, 4, \cdots, N-1) \end{aligned} \tag{3}$$

[Last Oscillator]

$$\begin{aligned} \dot{x}_N &= y_N \\ \dot{y}_N &= -x_N + \alpha(x_{N-1} - x_N) + \varepsilon\left(y_N - \frac{1}{3}y_N^3\right) \end{aligned} \tag{4}$$

where

$$t = \sqrt{L_1 C}\tau,\ i_{L_1 k} = \sqrt{\frac{C g_1}{3 L_1 g_3}} x_k,\ v_k = \sqrt{\frac{g_1}{3 g_3}} y_k,$$
$$\alpha = \frac{L_1}{L_0},\ \varepsilon = g_1\sqrt{\frac{L_1}{C}},\ \frac{d}{d\tau} = \text{“}\cdot\text{”} \tag{5}$$

It should be noted that α corresponds to the coupling and that ε corresponds to the nonlinearity. Equations (2)–(4) are calculated by using the fourth-order Runge-Kutta method.

3. Wave Propagation Phenomenon

In this section, wave propagation phenomenon observed from the circuit with 17 oscillators is investigated. Although we introduce the results only for 17 oscillators, the similar phenomena are observed from both of even and odd. Further, we could observe those from 100 oscillators.

Figure 2 shows typical examples of observed wave

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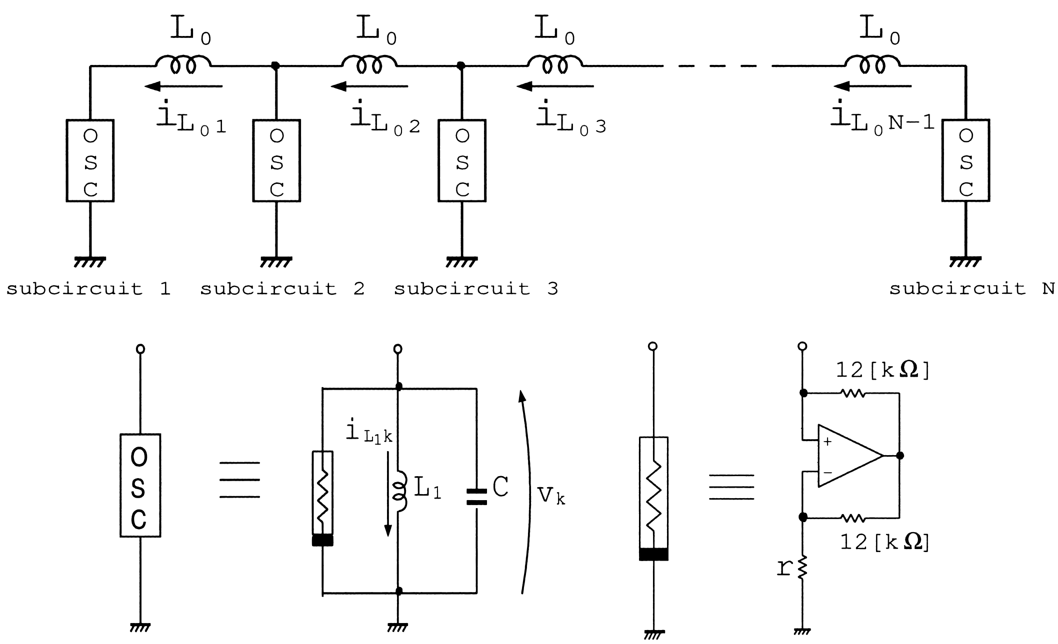


Fig. 1 Coupled van der Pol oscillators as a ladder.

propagation phenomena. These results are obtained for the same parameter values by changing initial conditions as follows: 1. Set initial conditions of all oscillators same. 2. Invert the sign of the voltages of one or two oscillators.

In upper diagrams, vertical axes are sum of voltages of adjacent oscillators and horizontal axes are time. Hence, the diagrams show how phase differences between adjacent oscillators change as time goes. White regions in the diagram correspond to the states that two adjacent oscillators are anti-phase synchronization. While, black regions correspond to the in-phase synchronization. In lower figures, snapshots of attractor of each oscillator and phase states between adjacent oscillators are shown.

In Fig. 2 (a) wave vanishes and phase states settle down to regular in-phase synchronization mode. On the other hand in other figures we can see that wave propagation phenomena continue to exist. As far as we know, such a continuously existing wave propagation phenomena of phase states in simple real circuits have never been reported. Further, we can see that there are two different scenario for the collision of two waves. Namely, in Fig. 2 (a) two waves extinct when they collides. While, in Fig. 2 (b) two waves reflect and the wave phenomenon continues to exist.

Next, we explain the mechanism of the generation of the wave by using the change of the oscillation frequencies according to the synchronization states. It has been already known that oscillation frequency of in-phase synchronization of oscillators coupled by inductors is different from that of anti-phase synchronization. Namely, f_{in}, oscillation frequency of in-phase synchronization, is smaller than f_{anti}, oscillation frequency of anti-phase synchronization. Further, the difference betweens f_{in} and f_{anti} increases as coupling inductance increases [6].

Throughout the paper, we define the phase difference between two adjacent oscillators and the frequency of OSCk as follows:

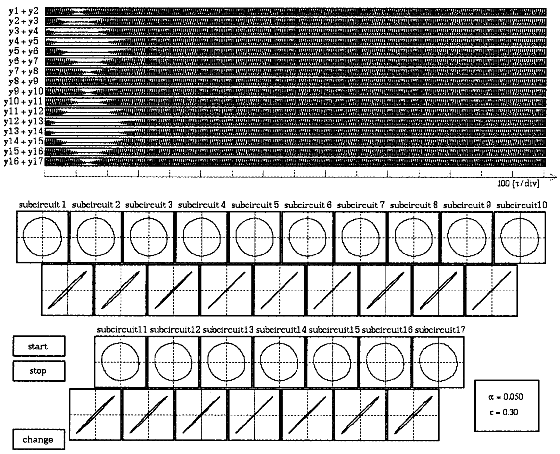
$$\Phi_{k,k+1}(n) = \frac{\tau_k(n) - \tau_{k+1}(n)}{\tau_k(n) - \tau_k(n-1)} \times \pi$$
$$f_k(n) = \frac{1}{2(\tau_k(n) - \tau_k(n-1))} \quad (6)$$

where $\tau_k(n)$ is time when the voltage of OSCk crosses 0[V] at n-th time.

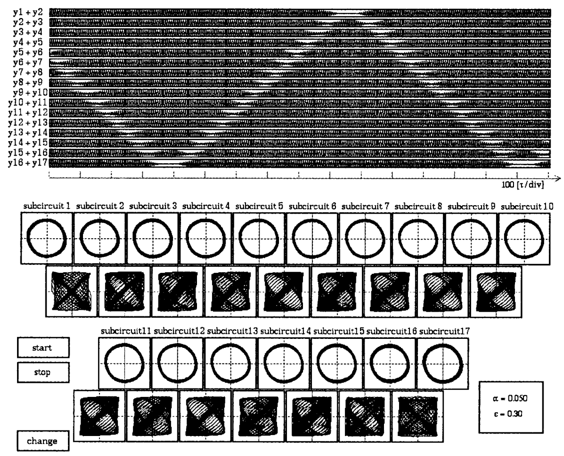
3.1 Mechanism of Wave Propagation

1. Let us assume that OSC1 – OSC6 are in-phase synchronization and that the wave changing from in-phase into anti-phase is going to reach OSC6 from the direction of OSC17. ($\tau = \tau_1$ in Fig. 3.)
2. As $\Phi_{6,7}$ approaches π, oscillation frequency of OSC6 f_6 changes from f_{in} to f_{anti}. ($\tau = \tau_2$ in Fig. 3.)
3. The change of oscillation frequency of OSC6 causes increase of phase difference between OSC5 and OSC6. Namely $\Phi_{5,6}$ increases. Speed of the increase is decided by the difference between f_5 and f_6. This means that propagation speed of the wave is decided by the difference between f_{in} and f_{anti}.
4. When $\Phi_{5,6}$ reaches almost π, f_6 is equal to f_{anti}. ($\tau = \tau_3$ in Fig. 3.)

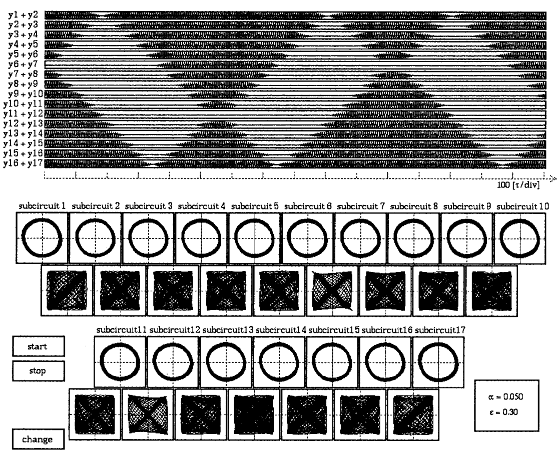
Although the above explanation is for the case of changing from in-phase to anti-phase, changing from anti-phase to in-phase can be explained in a similar



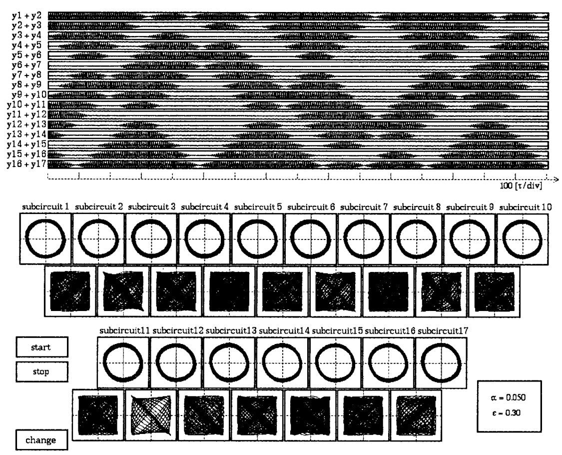
Wave-Motion (a)



Wave-Motion (b)



Wave-Motion (c)



Wave-Motion (d)

Fig. 2 Typical examples of wave propagation phenomena. $\alpha = 0.05$, $\varepsilon = 0.3$ and $\Delta\tau = 0.001$.

manner.

Figure 4 shows computer calculated results of phase differences and oscillation frequencies in the middle of the array. In the figure we can see two waves; one is the changing from in-phase to anti-phase and the other is the changing from anti-phase to in-phase.

3.2 Mechanism of Wave Reflection at the Edges of Array

1. Let us assume that only OSC1 and OSC2 are in-phase synchronization and that a wave from in-phase to anti-phase reaches at the edge.
2. Oscillation frequency of OSC2 f_2 begins to change from f_{in} toward f_{anti}, because in-phase synchronization between OSC2 and OSC3 breaks.
3. The change of f_2 causes slipping of $\Phi_{1,2}$.
4. Oscillation frequency of OSC1 f_1 also begins to change. However, it cannot reach f_{anti}, because OSC1 is at an edge and there are no effect from the other side. Hence, $\Phi_{1,2}$ continues to increase until reaching 2π.
5. By the effect of the decrease of ($\Phi_{1,2} \bmod 2\pi$), f_2 begins to decreases again from f_{anti} toward f_{in}.

Figure 5 shows computer calculated results of phase differences and oscillation frequencies at the edge of the array. In the figure we can see that a wave reflects as the manner explained above.

3.3 Mechanism of Wave Reflection at the Middle of Array

1. Let us assume that OSC7 – OSC10 are in-phase synchronization and that the waves changing from in-phase into anti-phase are going to reach OSC7 and OSC10 at the almost same time from the di-

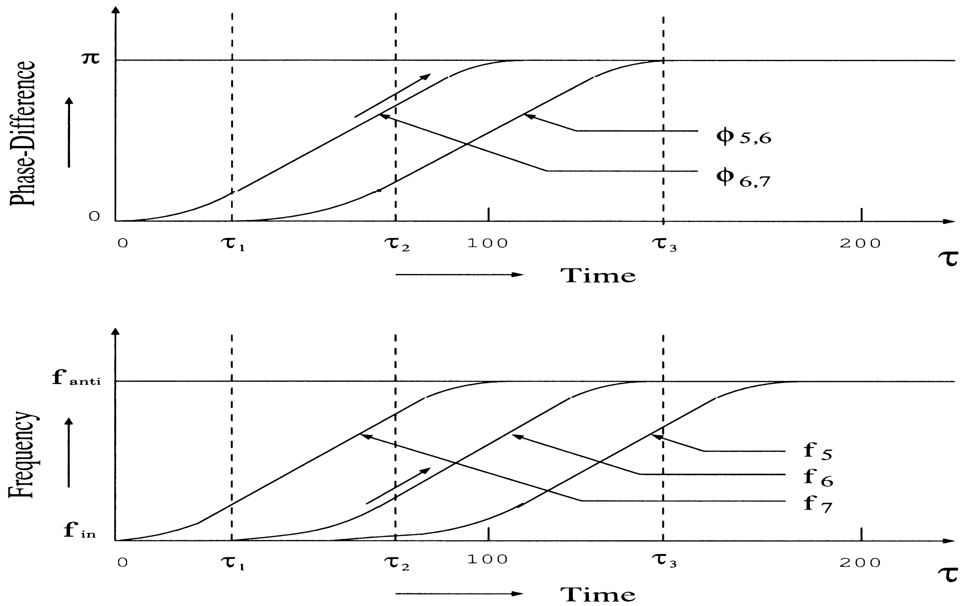


Fig. 3 Mechanism of wave propagation (outline charts).

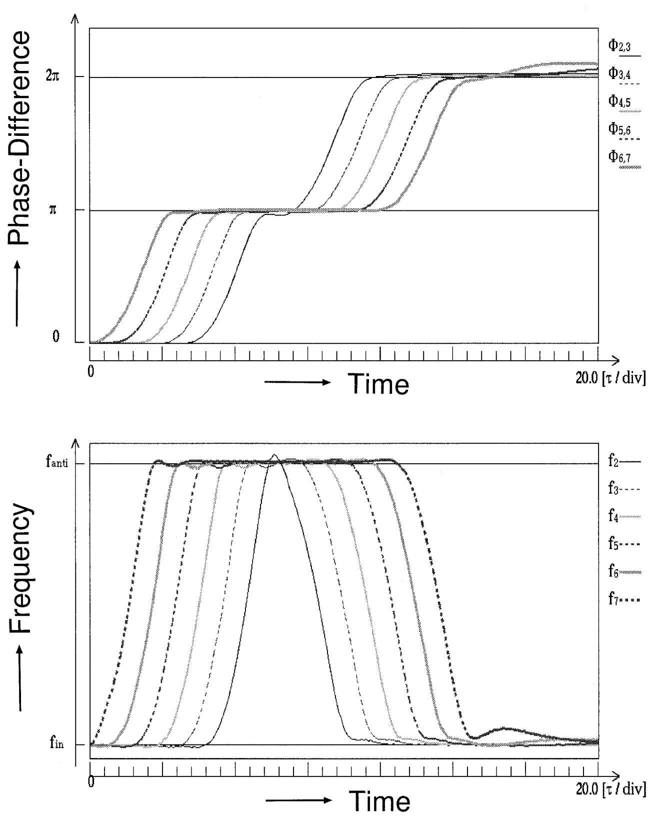


Fig. 4 Wave propagation (computer calculated results).

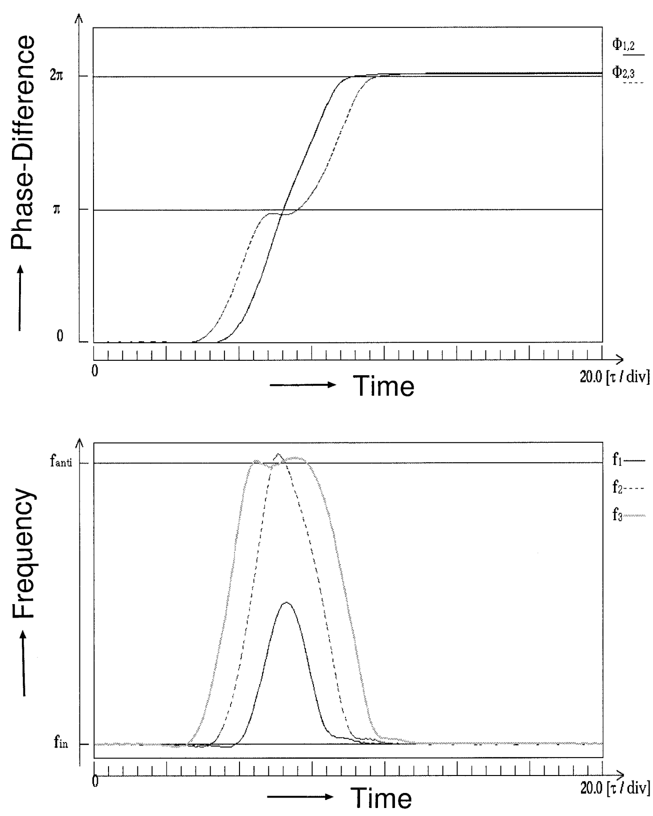


Fig. 5 Wave reflection at the edge of the array (computer calculated results).

rections of OSC1 and OSC17, respectively.

2. Waves reach OSC7 and OSC10 almost equal timing. $\Phi_{7,8}$ and $\Phi_{9,10}$ approach π and $-\pi$ by almost equal timing.
3. Because $\Phi_{7,8}$ and $\Phi_{9,10}$ approach π and $-\pi$ respectively, oscillation frequencies of OSC8 f_8 and OSC9 f_9 change from f_{in} toward f_{anti}.
4. However because f_8 and f_9 change almost simultaneously, in-phase synchronization between OSC8

and OSC9 does not break. Namely, $\Phi_{8,9}$ remains almost 0. Hence, f_8 and f_9 do not reach f_{anti}. Accordingly $\Phi_{7,8}$ and $\Phi_{9,10}$ continue to change until reaching 2π and -2π, respectively.

5. By the effect of the decreases of ($\Phi_{7,8}$ mod 2π) and ($\Phi_{9,10}$ mod 2π), f_8 and f_9 begin to decrease toward f_{in} again.

Waves changing from anti-phase to in-phase can be explained in a similar manner.

Figure 6 shows computer calculated results of phase differences and oscillation frequencies in the middle of the array. In the figure we can see that a wave reflects as the manner explained above.

3.4 Mechanism of Wave Extinction

1. Let us assume that OSC5 – OSC7 are anti-phase synchronization and that the waves changing from anti-phase into in-phase are going to reach OSC5 and OSC7 at the almost same time from the directions of OSC1 and OSC17 respectively.
2. Reaching the waves causes the changes of oscillation frequencies of OSC5 f_5 and OSC7 f_7 changes from f_{anti} to f_{in}.
3. Accordingly $\Phi_{5,6}$ and $\Phi_{6,7}$ change from π to 0 and 2π respectively.
4. Changes of $\Phi_{5,6}$ and $\Phi_{6,7}$ cause that the change of oscillation frequency of OSC6 f_6 from f_{anti} to f_{in}.
5. When $\Phi_{5,6}$ and $\Phi_{6,7}$ reach 2π and 0 respectively, f_6 become to be equal to f_{in}. The whole array results in stable in-phase synchronization.

Waves changing from in-phase to anti-phase can be explained in a similar manner.

Figure 7 shows computer calculated results of phase differences and oscillation frequencies in the middle of the array. In the figure we can see that a wave extinction as the manner explained above.

According to the above-explained mechanisms of the wave reflection and the wave extinction in the middle of the array, we can conclude that the two waves colliding in the middle of the array will;

1. Reflect when the waves reach OSCk and OSC$k+1$ at the almost equal timing.
2. Extinct when the waves reach OSC$k-1$ and OSC$k+1$ at the almost equal timing.

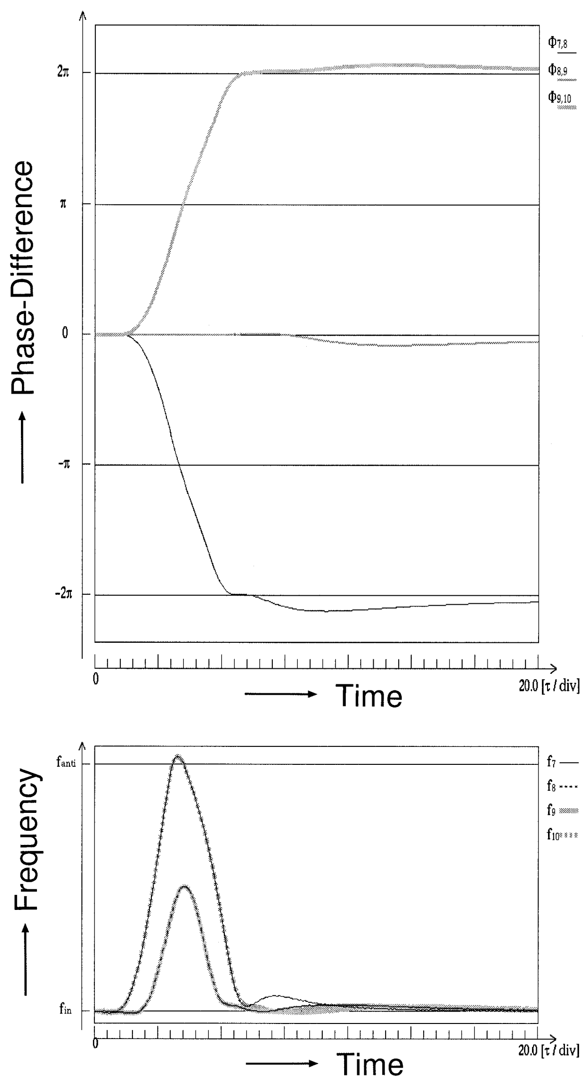


Fig. 6 Wave reflection at the middle array (computer calculated results).

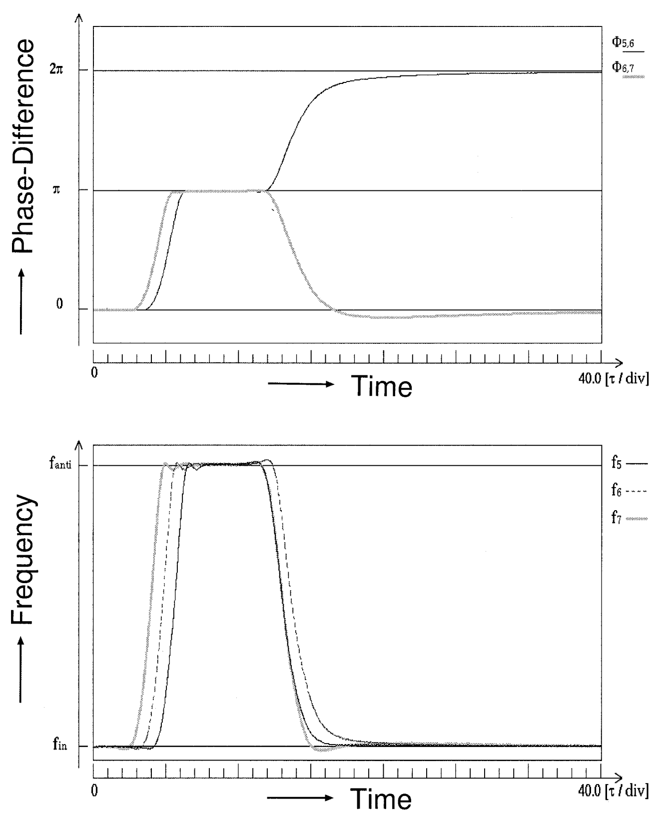
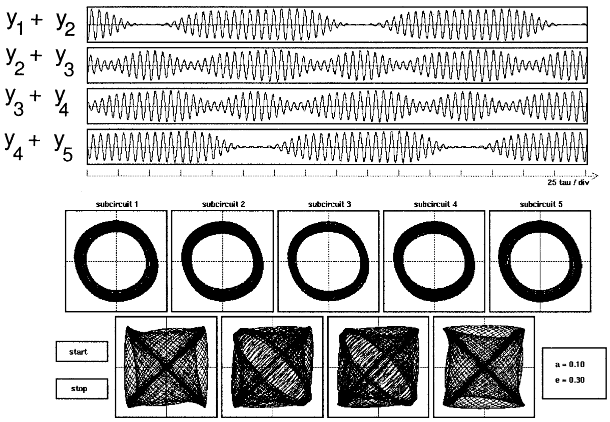
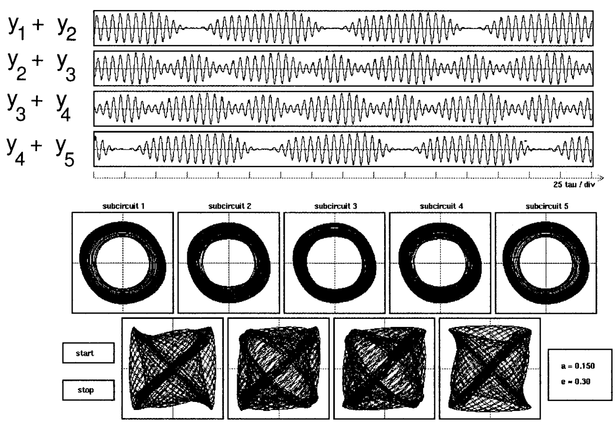


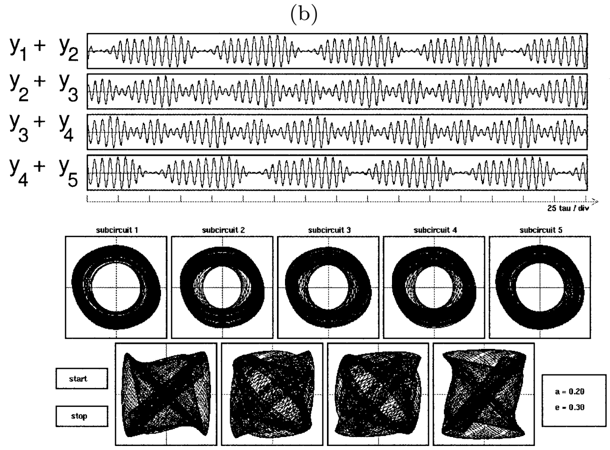
Fig. 7 Wave extinction (computer calculated results).



(a)



(b)



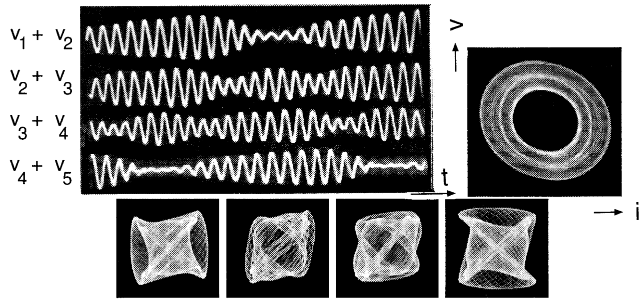
(c)

Fig. 8 Computer calculated results ($N = 5$). (a) $\alpha = 0.10$, (b) $\alpha = 0.15$, (c) $\alpha = 0.2$.

4. Circuit Experiments

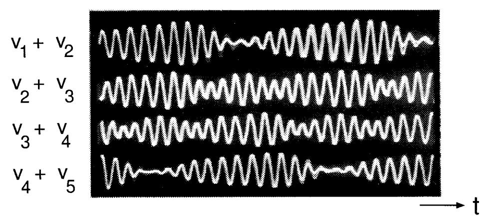
In this section circuit experimental results are shown.

Because it is difficult to carry out circuit experiments for the array with the size of $N = 17$, we carried out both computer calculations and circuit experiments for the array with the size of $N = 5$.

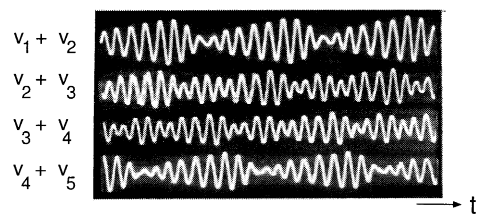


(i) (ii) (iii) (iv)

(a)



(b)



(c)

Fig. 9 Circuit experimental results ($N = 5$). (a) $L_0 = 1200\,\text{mH}$, (b) $L_0 = 950\,\text{mH}$, (c) $L_0 = 650\,\text{mH}$. (i) v_1 vs. v_2, (ii) v_2 vs. v_3, (iii) v_3 vs. v_4, (iv) v_4 vs. v_5.

Figure 8 shows computer calculated results for $\varepsilon = 0.30$ and $\Delta\tau = 0.01$. We can see that wave propagation phenomenon appears even in this small number of oscillators case. Further, we can confirm that wave propagation speed increases as coupling parameter increases.

Figure 9 shows circuit experimental results for $L_1 = 200\,\text{mH}$, $C = 100\,\text{nF}$ and $r = 1.0\,\text{k}\Omega$. Circuit experimental results agree well with computer calculated results qualitatively.

The phenomena correponding to Figs. 2 (c) and (d) are not observed in circuit experiments. We consider that this is because the number of elements is small. However, detailed investigation is our future study.

5. Conclusions

In this study, we investigated wave propagation phenomena of phase states observed from van der Pol oscillators coupled by inductors as a ladder. For the case of 17 oscillators, we found interesting wave propagation phenomena of phase states. By using the relationship between phase states and oscillation frequencies, we explained the mechanisms of the propagation and the reflection of wave. Circuit experimental results agreed well with computer calculated results qualitatively.

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