PAPER Special Section on Nonlinear Theory and its Applications

Synchronization Phenomena in Resistively Coupled Oscillators with Different Frequencies

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SUMMARY In this study, some oscillators with different oscillation frequencies, $N-1$ oscillators have the same oscillation frequency and only the Nth oscillator has different frequency, coupled by a resistor are investigated. At first we consider non-resonance. By carrying out circuit experiments and computer calculations, we observe that oscillation of the Nth oscillator stops in some range of the frequency ratio and that others are synchronized as if the Nth oscillator does not exist. These phenomena are also analyzed theoretically by using the averaging method. Secondly, we investigate the resonance region where the frequency ratio is nearly equal to 1. For this region we can observe interesting double-mode oscillation, that is, synchronization of envelopes of the double-mode oscillation and change of oscillation amplitude of the Nth oscillator.

key words: *coupled oscillator, synchronization, double-mode oscillation*

1. Introduction

Since coupled oscillators systems can describe various phenomena in natural field, there have been many investigations on such systems and various interesting phenomena have reported by several researchers [1]–[5]. Kimura et al. investigated synchronization phenomena observed in two oscillators coupled by a resistor shown in Fig. 1 [1]. They confirmed that these oscillators were synchronized at the opposite phase. They also commented that the synchronization seemed to occur so that the system tends to minimize the energy consumed by the coupling resistor. This study was also extended at the case of three oscillators and they confirmed the generation of three phase synchronization [2]. Moreover, we have investigated many oscillators coupled by resistors [4]. In [4], we confirmed that for the case of four oscillators the system is synchronized at opposite phase in pairs, and the phase shift between two pairs is not decided. However, these studies treat only the case that all of oscillators have the same oscillation frequency. Of cause, there may be many systems modeled as coupled oscillators with the same oscillation frequency. However, because there are very few coupled systems with the exactly same oscillation frequency in natural field, various synchronization phenomena which cannot be explained by coupled systems with the same frequency exist. Although some studies treating coupled oscillators systems with different oscillation frequencies have been reported, we consider that many unknown nonlinear phenomena remain in such systems. Therefore, it is very important to investigate such systems.

In this study, we consider some oscillators with different oscillation frequencies coupled by a resistor. Because it is difficult to treat general case, we concentrate the simplest case that oscillation frequency of only one oscillator is varied. Namely, $N-1$ oscillators have the same oscillation frequency and only the Nth oscillator has different frequency. At first we consider non-resonance. By carrying out circuit experiments and computer calculations, we observe that oscillation of the Nth oscillator stops in some range of the frequency ratio and that others are synchronized as if the Nth oscillator does not exist. These phenomena are also analyzed theoretically by using the averaging method. Secondly, we investigate the resonance region where the frequency ratio is nearly equal to 1. For this region we can observe interesting double-mode oscillation, that is, synchronization of envelopes of the double-mode oscillation and change of oscillation amplitude of the Nth oscillator. Because the averaging method cannot be applied to the double-mode oscillation, these phenomena are investigated by both of circuit experiments and computer calculations in detail.

The results in this study mean that we can control the phase states of many oscillators by varying the oscillation frequency of only a few oscillators. Hence, the present study would contribute to the development of coupled oscillators networks [5] which are expected as one of future parallel information processing architectures.

2. Circuit Model

The circuit model is shown in Fig. 2. In the system only the Nth oscillator has different oscillation frequency. v_k is the voltage across the capacitor C and i_k is the cur-

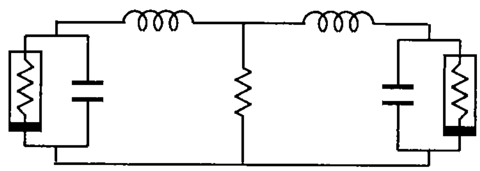

Fig. 1 Two oscillators coupled by a resistor.

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rent through the inductor L of kth oscillator. At first, we assume that $v-i$ characteristics of the nonlinear resistor in each oscillator is represented by the following third order polynomial equation.

$$i_r(v_k) = -g_1 v_k + g_3 v_k^3 \tag{1}$$

The circuit equation is given as follows.

$$C\frac{dv_k}{dt} = -i_k - i_r(v_k) \tag{2}$$
$$(k = 1, 2, \cdots, N-1)$$
$$\alpha C\frac{dv_N}{dt} = -i_N - i_r(v_N) \tag{3}$$
$$L\frac{di_k}{dt} = -v_k - R\sum_{j=1}^{N} i_j \tag{4}$$
$$(k = 1, 2, \cdots, N)$$

Changing the variables

$$t = \sqrt{LC}\tau, \quad \alpha = \frac{1}{\omega^2},$$
$$v_k = \sqrt{\frac{g_1}{3g_3}}x_k, \quad i_k = \sqrt{\frac{Cg_1}{3Lg_3}}y_k, \tag{5}$$

and defining

$$\beta = R\sqrt{\frac{C}{L}}, \quad \epsilon = g_1\sqrt{\frac{L}{C}}, \tag{6}$$

then the normalized equations are represented as follows

$$\dot{x}_k = \epsilon\left(x_k - \frac{1}{3}x_k^3\right) - y_k \tag{7}$$
$$(k = 1, 2, \cdots, N-1)$$
$$\dot{x}_N = \omega^2\epsilon\left(x_N - \frac{1}{3}x_N^3\right) - \omega^2 y_N \tag{8}$$
$$\dot{y}_k = x_k - \beta\sum_{j=1}^{N} y_j \quad (k = 1, 2, \cdots, N) \tag{9}$$

where β is the coupling factor and ϵ is the largeness of

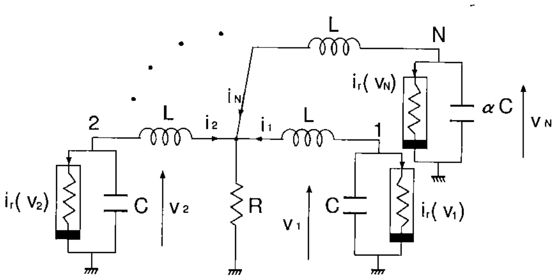

Fig. 2 N oscillators coupled by a resistor with one different frequency oscillator.

nonlinearity. Eqs. (7)–(9) may be combined to give the second order nonliner differential equation as follows

$$\ddot{x}_k + x_k = \epsilon(1 - x_k^2)\dot{x}_k + \beta Y \equiv F_k \tag{10}$$
$$(k = 1, 2, \cdots, N-1)$$
$$\ddot{x}_N + \omega^2 x_N = \omega^2\epsilon(1 - x_k^2)\dot{x}_N + \omega^2\beta Y$$
$$\equiv \omega^2 F_N \tag{11}$$
$$\dot{Y} + N\beta Y = \sum_{j=1}^{N} x_j \quad (k = 1, 2, \cdots, N) \tag{12}$$

where

$$Y \equiv \sum_{j=1}^{N} y_k \tag{13}$$

3. Circuit Experiments and Numerical Calculations

We carried out circuit experiments and numerical calculations for the case of $N = 3$ and $N = 4$. For computer calculations Eqs. (7)–(9) are calculated by using the Runge-Kutta method with step size $\Delta\tau = 0.125$.

Figure 3 shows observed phenomena for $N = 3$. For the case that all of these oscillators have the same

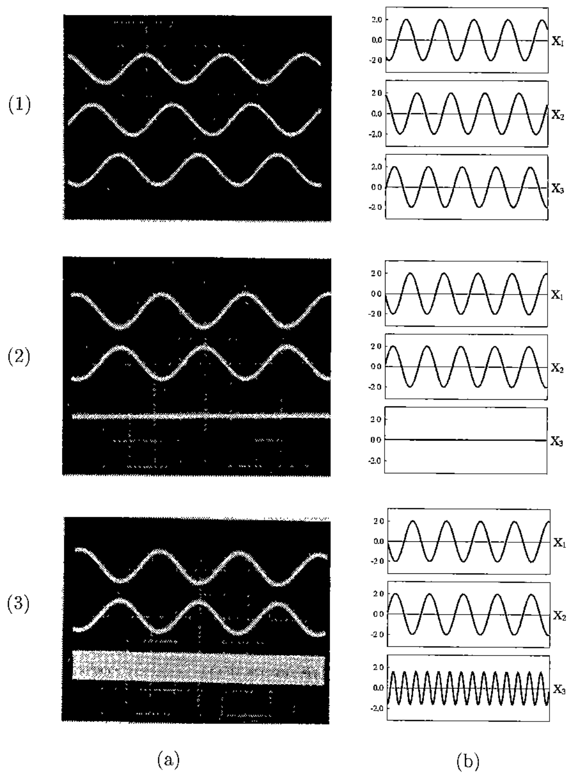

Fig. 3 Experimental results for $N = 3$.
(a) Circuit experiments for L = 10[mH], C = 100[nF], R = 5.6[kΩ]
(b) Computer culculation for $\beta = 0.1, \epsilon = 0.03$
(1) Three-phase synchronization for $\omega = 1.00$.
(2) Opposite-phase synchronization and oscillation death for $\omega = 1.64$.
(3) Opposite-phase synchronization and independent oscillations for $\omega = 4.47$.

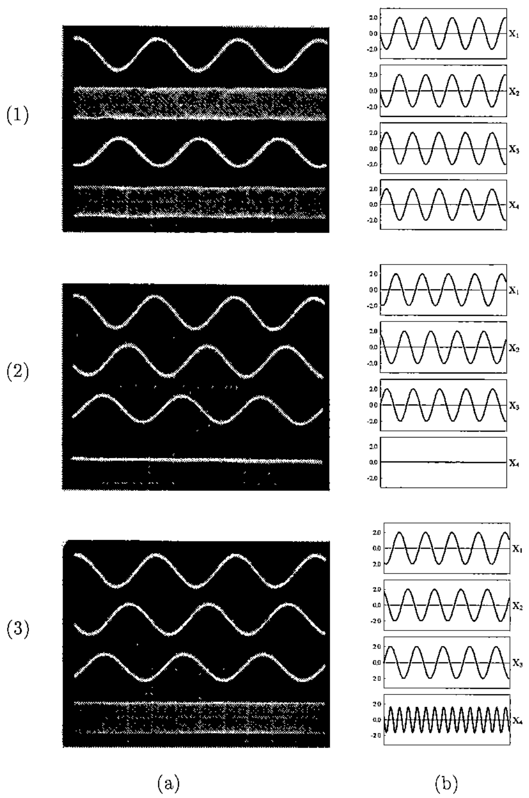

Fig. 4 Experimental results for $N = 4$.
(a) Circuit experiments for L = 10[mH], C = 100[nF], R = 5.6[kΩ]
(b) Computer culculation for $\beta = 0.1, \epsilon = 0.03$
(1) Opposite-phase synchronization in pairs for $\omega = 1.00$.
(2) Three-phase synchronization and oscillation death for $\omega = 1.64$.
(3) Three-phase synchronization and independent oscillations for $\omega = 4.47$.

frequency (Fig. 3 (1)), we can observe that the system is synchronized at the three-phase. This phenomenon had been confirmed and analyzed theoretically in [2]. When the frequency of the 3rd oscillator is varied, we observed that oscillation of the 3rd oscillator stops in some range of the frequency, namely oscillation death appears. And the others are synchronized at the opposite-phase (Fig. 3 (2)). As increasing the frequency of the 3rd oscillator, oscillation of the 3rd oscillator starts again (Fig. 3 (3)). However, the 3rd oscillator is not synchronized to the others. Namely, the 3rd oscillator oscillates alone and the others keep opposite-phase synchronization.

Figure 4 shows observed phenomena for $N = 4$. We can observe similar synchronization phenomena. When the frequencies of the four oscillators are the same, the system is synchronized at the opposite-phase in pairs and the phase shift between two pairs is not decided (Fig. 4 (1)). Similar phenomenon had been confirmed in [3], [4]. When the frequency of the 4th oscillator is varied, oscillation of the 4th oscillator stops, namely oscillation death appears. And the others are synchronized at the three-phase (Fig. 4 (2)). As increasing the frequency of the 4th oscillator, the 4th oscillator oscillates alone and the others are synchronized at the three-phase (Fig. 4 (3)).

Moreover, we observed interesting double-mode oscillation in the resonance region of $\omega \simeq 1$ for both case of $N = 3$ and 4. We will explain this phenomenon later.

4. Theoretical Analysis

In this section we analyze the synchronization phenomena in the previous section by using the averaging method.

Equation (12) is first order linear differential equation. The solution is given as follows

$$Y = e^{-3\beta\tau} \int e^{3\beta\tau} \sum_{j=1}^{N} x_j d\tau + Ce^{-3\beta\tau} \quad (14)$$

(C : const.)

In the steady state, the second term of Eq. (14) becomes to zero. Let us assume the solutions of Eqs. (10)–(11) are

$$x_k(\tau) = \rho_k \cos(\tau + \theta_k) \quad (15)$$

$$(k = 1, 2, \cdots, N-1)$$

$$x_N(\tau) = \rho_N \cos(\omega\tau + \theta_N). \quad (16)$$

We pay attention to treat the non-resonance system and apply for the averaging method to Eqs. (10)-(11). We obtain

$$\dot{\rho}_k = -\frac{\epsilon\rho_k}{8}(\rho_k^2 - 4) - \frac{\beta}{9\beta^2+1}\left\{\frac{3\beta}{2}\sum_{j=1}^{N-1}\rho_j \sin(\theta_k - \theta_j) + \frac{1}{2}\sum_{j=1}^{N-1}\rho_j \cos(\theta_k - \theta_j)\right\} \quad (17)$$

$$(k = 1, 2, \cdots, N-1)$$

$$\dot{\rho}_N = \frac{\omega^3\epsilon\rho_N}{8} \times \left\{\rho_N^2 - 4\left(1 - \frac{\beta}{\epsilon(9\beta^2+\omega^2)}\right)\right\} \quad (18)$$

$$\dot{\theta}_k = -\frac{\beta}{9\beta^2+1}\left\{\frac{3\beta}{2}\sum_{j=1}^{N-1}\rho_j \cos(\theta_k - \theta_j) + \frac{1}{2}\sum_{j=1}^{N-1}\rho_j \sin(\theta_k - \theta_j)\right\} \quad (19)$$

$$(k = 1, 2, \cdots, N-1)$$

$$\dot{\theta}_N = \frac{3\beta^2\omega^2\rho_N}{2(9\beta^2+\omega^2)} = C \quad \text{(C : const.)} \quad (20)$$

In the steady state

$$\dot{\rho}_k = 0 \text{ and } \dot{\theta}_k = 0 \quad (21)$$

$$(k = 1, 2, \cdots, N)$$

must be satisfied regardless of the value of β. In addition, if ρ_N is not zero, $\dot{\theta}_N$ cannot be zero. This means that Nth oscillator is not synchronized to the others. We obtain the solutions as follows

- For the case of $N = 3$

$$\rho_1 = \rho_2 = 2, \quad \theta_1 - \theta_2 = \pi \tag{22}$$

$$\rho_3 = \begin{cases} 2\sqrt{1 - \dfrac{\beta}{\epsilon\,(9\beta^2 + \omega^2)}} & \cdots 1 - \dfrac{\beta}{\epsilon\,(9\beta^2 + \omega^2)} > 0 \\ 0 & \cdots 1 - \dfrac{\beta}{\epsilon\,(9\beta^2 + \omega^2)} < 0 \end{cases} \tag{23}$$

$$\theta_3 = At + B \quad (A, B : \text{ const.}) \tag{24}$$

- For the case of $N = 4$

$$\rho_1 = \rho_2 = \rho_3 = 2, \quad \theta_1 - \theta_2 = \frac{3}{2}\pi,$$

$$\theta_1 - \theta_3 = -\frac{3}{2}\pi \tag{25}$$

$$\rho_4 = \begin{cases} 2\sqrt{1 - \dfrac{\beta}{\epsilon\,(9\beta^2 + \omega^2)}} & \cdots 1 - \dfrac{\beta}{\epsilon\,(9\beta^2 + \omega^2)} > 0 \\ 0 & \cdots 1 - \dfrac{\beta}{\epsilon\,(9\beta^2 + \omega^2)} < 0 \end{cases} \tag{26}$$

$$\theta_4 = At + B \quad (A, B : \text{ const.}) \tag{27}$$

In this system when N is more than 4 we can't observe the synchronization without increasing the nonlinearlty ϵ. If the nonlinearlty is increased, we can't treat the system using the averaging method.

For the case of $N = 3$, Eq. (22) shows that the oscillators 1 and 2 are synchronized at the opposite-phase. Eq. (23) shows that the amplitude of the 3rd oscillator is a function of ω and becomes zero for some range of ω. Eq. (24) shows that the 3rd oscillator does not synchronize to the others.

For the case of $N = 4$, Eq. (25) shows that the oscillators 1,2 and 3 are synchronized at the three-phase. Eq. (26) shows that the amplitude of the 4th oscillator is a function of ω and becomes zero for some range of ω. Eq. (27) shows that the 4th oscillator does not synchronize to the others.

Using Eq. (23) the relationship between the amplitude of the Nth oscillator ρ_N and the frequency ratio ω is shown in Fig. 5. In the figure, we show the numerical results together. We can see both results agree well except for resonance region.

For the non-resonance case, we can consider that this system is completely separated into the oscillator with the different frequency and the others. Therefor, the stability problem of the solutions in Eqs. (22) (25) is equal to [1], [2]. While the stability of Eqs. (23) (26) are easily confirmed by substituting Eqs. (23) (26) into the equation given by partial differential Eq. (18) with respect to ρ_N.

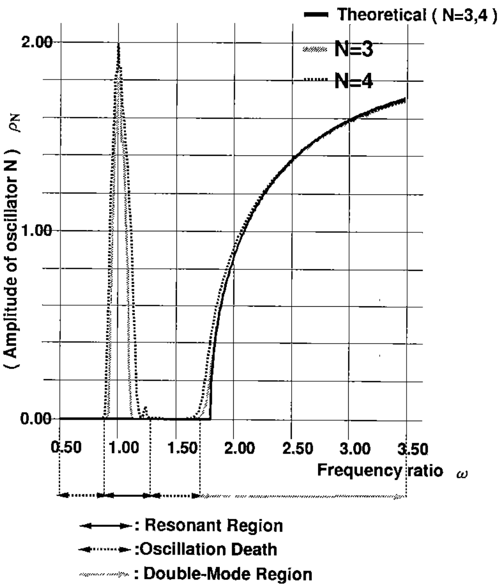

Fig. 5 Relationship between ρ_3, ρ_4 and ω for $\beta = 0.1$ and $\epsilon = 0.03$.

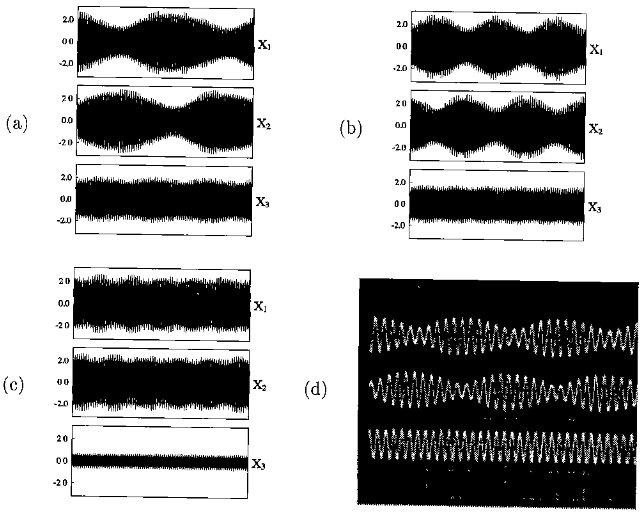

Fig. 6 Double-mode oscillation in resonance region for $N = 3$ ($\beta = 0.1, \epsilon = 0.03$).
(a) Computer culculation for $\omega = 1.03$.
(b) Computer culculation for $\omega = 1.05$.
(c) Computer culculatin for $\omega = 1.09$.
(d) Circuit experimental result for L=10[mH], C=74[nF] and R=5.6[kΩ].

5. Resonance Region

In this section we consider the resonance region, namely $\omega \simeq 1$. Experimental and computer calculator results

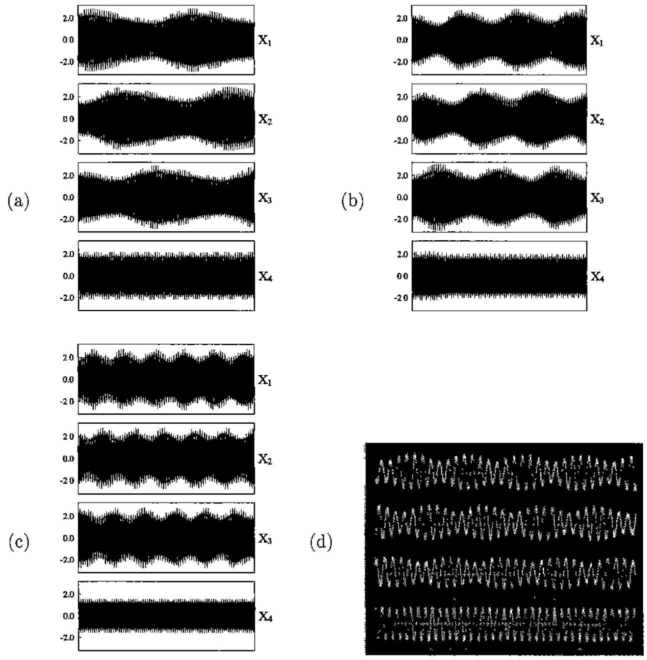

Fig. 7 Double-mode oscillation in resonance region for $N = 4$ ($\beta = 0.1, \epsilon = 0.03$).
(a) Computer culculation for $\omega = 1.03$.
(b) Computer culculation for $\omega = 1.05$.
(c) Computer culculatin for $\omega = 1.09$.
(d) Circuit experimental result for L=10[mH], C=74[nF] and R=5.6[kΩ].

for $N=3$ are shown in Fig. 6. From the figures we can observe the generation of interesting double-mode oscillation.

Namely, oscillators 1 and 2 generate double-mode oscillation and its envelopes are synchronized at the opposite-phase. While the oscillation of the 3rd oscillator seems to have only one oscillation frequency, if we neglect small effect. As ω increases, we can observe that the beat frequency of the double-mode oscillation becomes larger and that the amplitude of the 3rd oscillator becomes small and vanishes.

The observed results for $N = 4$ are shown in Fig. 7. We can see that the envelopes of the double-mode oscillation are synchronized at the three-phase. The changes of beat frequency and the amplitude of 4th oscillator are also confirmed from the figure.

Because the averaging method cannot be applied to the double-mode oscillation, we cannot analyze these phenomena theoretically. However, we would like to emphasize that these phenomena are observed from both of circuit experiments and computer calculations.

6. Conclusions

In this study, some oscillators with different oscillation frequencies, $N-1$ oscillators have the same oscillation frequency and only the Nth oscillator has different frequency, coupled by a resistor have been investigated. At first we consider non-resonance. By carrying out circuit experiments and computer calculations, we observed that oscillation of the Nth oscillator stops in some range of the frequency ratio and that others are synchronized as if the Nth oscillator does not exist. These phenomena are also analyzed theoretically by using the averaging method. Secondly, we investigate the resonance region where the frequency ratio is nearly equal to 1. For this region we can observe interesting double-mode oscillation, that is, synchronization of envelopes of the double-mode oscillation and change of oscillation amplitude of the Nth oscillator.

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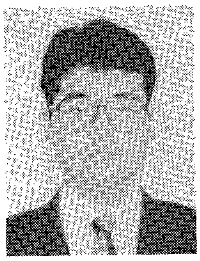

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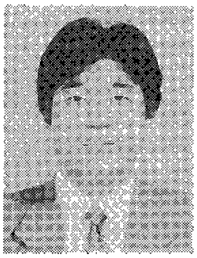

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