PAPER

Multimode Chaos in Two Coupled Chaotic Oscillators with Hard Nonlinearities

Yoshifumi NISHIO† *and* **Akio USHIDA**†, ***Members***

SUMMARY In this study, multimode chaos observed from two coupled chaotic oscillators with hard nonlinearities is investigated. At first, a simple chaotic oscillator with hard nonlinearities is realized. It is confirmed that in this chaotic oscillator the origin is always asymptotically stable and that the solution, which is excited by giving relatively large initial conditions, undergoes period-doubling bifurcations and bifurcates to chaos. Next, the coexistence of four different modes of oscillations are observed from two coupled chaotic oscillators with hard nonlinearities by both of circuit experiments and computer calculations. One of the modes of oscillation is a nonresonant double-mode oscillation and this oscillation is stably generated even in the case that oscillation is chaotic. Namely, for this oscillation mode, chaotic oscillation and periodic oscillation can be simultaneously excited. This phenomenon has not been reported yet, and we name this phenomenon as double-mode chaos. Finally, the beat frequency of the double-mode chaos is confirmed to be changed by varying the value of the coupling capacitor.
key words: *chaotic circuit, hard nonlinearity, coupled oscillator, multimode oscillation*

1. Introduction

Coupled oscillators systems are good model to describe various nonlinear phenomena in the field of natural science and a number of excellent studies on mutual synchronization of oscillators have been carried out (e.g. [1]–[4]). Oscillators containing a nonlinear resistor whose $v - i$ characteristics are described by fifth-power nonlinear characteristics are known to exhibit hard excitation [5], [6]. Namely, the origin is asymptotically stable and an proper initial condition, which is larger than a critical value, is necessary to generate the oscillation. Such an oscillator is often called as hard oscillator or said to have hard nonlinearity. Datardina and Linkens have investigated two identical oscillators with hard nonlinearities coupled by a inductor as shown in Fig. 1 (a) or equivalently Fig. 1 (b) [2]. They have confirmed that nonresonant double-mode oscillations, which could not occur for the case of third-power nonlinearity, were stably excited in the coupled system. They have also confirmed that four different modes coexist for some range of parameter values; zero, two single-modes, and a double-mode. Recently, Yoshinaga and Kawakami have investigated the double-mode oscillation observed from an arbitrary number of identical oscillators with hard nonlinearities coupled by inductors as a ring [4]. They confirmed the envelopes of nonresonant double-mode oscillations were synchronized. As these studies, coupled systems of oscillators with hard nonlinearities exhibit interesting synchronization phenomena of nonresonant double-mode oscillations as well as single-modes.

On the other hand, many nonlinear dynamical systems in the various fields have been clarified to exhibits chaotic oscillations and recently applications of chaos to engineering systems attract many researchers' attentions [7]. Among the studies on such applications, synchronization of chaotic systems or signals is significant because such a technique is necessary to realize communication systems using chaos. Since the chaotic solution is unstable and small error of initial values are expanded as time goes, the synchronization of chaos seems to be extremely interesting phenomena. Further, because coupled chaotic oscillators are good example of higher-dimensional systems which have been also drawing recent attentions, investigating what kind of phenomena are observed from such systems would contribute to develop the study of nonlinear dynamical systems. We have already studied synchronization phenomena observed from some types of coupled chaotic oscillators [8]–[10]. In the studies it has been confirmed that quasi-synchronization occurred even if each oscillator exhibits chaos. However, since we did not consider oscillators with hard nonlinearities in the previous

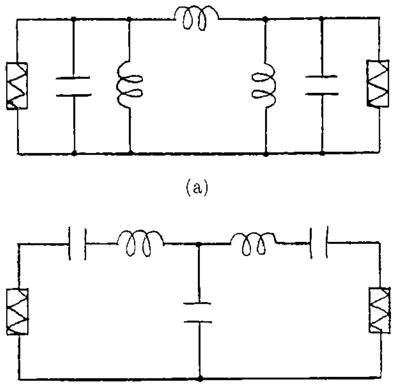

Fig. 1 (a) Coupled oscillators studied in Ref. [2]. (b) Equivalent circuit.

Manuscript received July 10, 1995.
Manuscript revised October 16, 1995.
†The authors are with the Faculty of Engineering, Tokushima University, Tokushima-shi, 770 Japan.

studies, we have not obtained any results about double-mode oscillations.

In this study, we investigate multimode chaos observed from two coupled chaotic oscillators with hard nonlinearities. At first, a simple chaotic oscillator with hard nonlinearities is realized. It is confirmed that in this chaotic oscillator the origin is always asymptotically stable and that the solution, which is excited by giving relatively large initial conditions, undergoes period-doubling bifurcations and bifurcates to chaos. Next, the coexistence of four different modes of oscillations are observed from two coupled chaotic oscillators with hard nonlinearities by both of circuit experiments and computer calculations. One of the modes of oscillation is a nonresonant double-mode oscillation and this oscillation is stably generated even in the case that oscillation is chaotic. Namely, for this oscillation mode, chaotic oscillation and periodic oscillation can be simultaneously excited. This phenomenon has not been reported yet, and we name this phenomenon as double-mode chaos. Finally, the beat frequency of the double-mode chaos is confirmed to be changed by varying the value of the coupling capacitor.

2. Chaotic Oscillator with Hard Nonlinearity

Figure 2 shows the realization of chaotic oscillator with hard nonlinearity. If we remove a resistor R_d and a pair of diodes connected in parallel with R_d, the circuit is the symmetric version of the chaotic circuit proposed by Inaba et al. [11].

At first, we approximate the $i-v$ characteristics of the diodes in the circuit as two-segment piecewise linear function as shown in Fig. 3.

$$v_d(i) = \begin{cases} r_d i & \cdots \quad (i \leqq V/r_d) \\ V & \cdots \quad (i > V/r_d). \end{cases} \tag{1}$$

In this case, the $i-v$ characteristics of the nonlinear resistor including a linear negative resistor and of the nonlinear resistor consisting of six diodes are described by three-segment piecewise linear functions as shown in Fig. 4.

$$v_r(i) = \begin{cases} V - ri & \cdots \quad (i > J) \\ \left(\dfrac{R_d r_d}{2R_d + r_d} - r\right) i & \cdots \quad (|i| \leqq J) \\ -V - ri & \cdots \quad (i < -J), \end{cases} \tag{2}$$

$$\left(\text{where} \quad J = \frac{2R_d + r_d}{R_d r_d} V \right),$$

$$v_D(i) = \begin{cases} 3V & \cdots \quad (i > 2V/r_d) \\ \dfrac{3}{2} r_d(i) & \cdots \quad (|i| \leqq 2V/r_d) \\ -3V & \cdots \quad (i < -2V/r_d). \end{cases} \tag{3}$$

The equation governing the circuit in Fig. 2 is described as follows:

$$L_1 \frac{di}{dt} = -v - Ri - v_D(i + I)$$

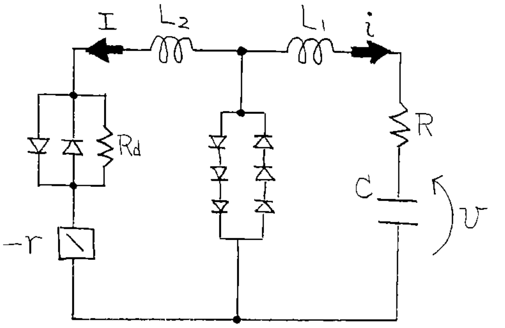

Fig. 2 Realization of chaotic oscillator with hard nonlinearity.

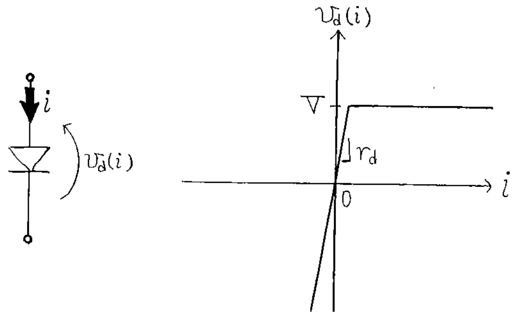

Fig. 3 Approximation of the $i-v$ characteristics of the diode.

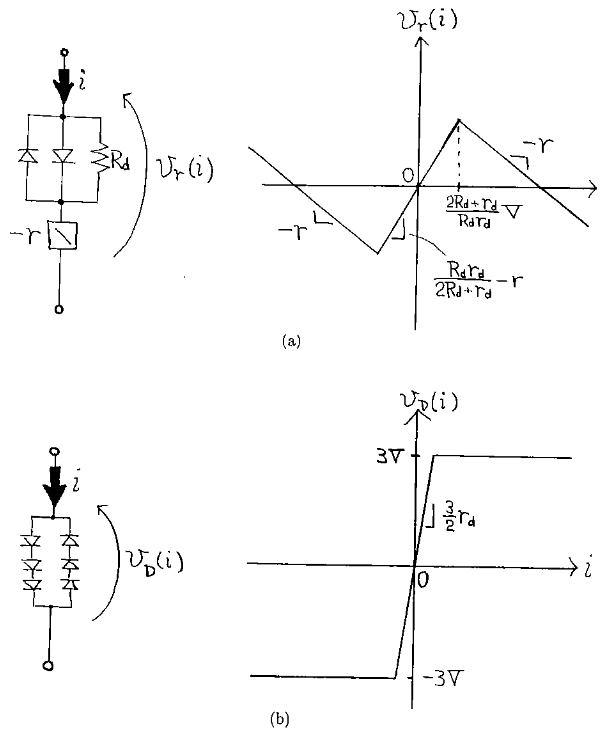

Fig. 4 The $i-v$ characteristics of (a) the nonlinear resistor including a linear negative resistor and (b) the nonlinear resistor consisting of six diodes.

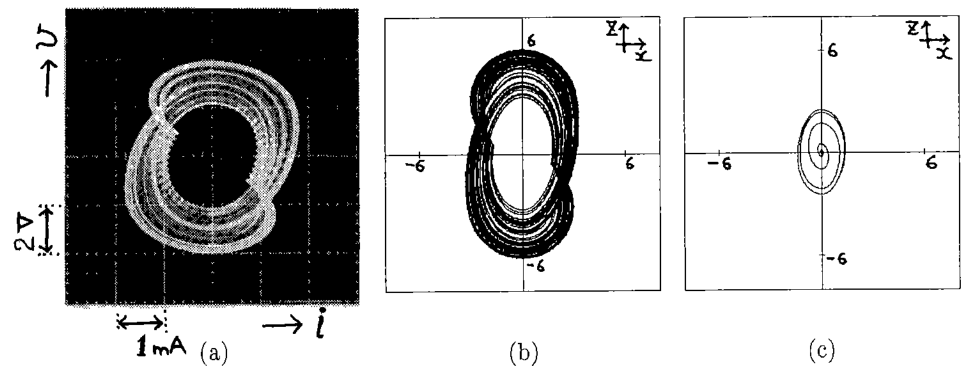

Fig. 5 (a) Typical example of chaotic attractors (L_1 = 100 mH, L_2 = 200 mH, C = 0.068 μF, $R = 50\,\Omega$, $R_d = 2\,\mathrm{k}\Omega$ and $r = 1.47\,\mathrm{k}\Omega$). (b) Corresponding computer calculation ($\beta = 0.5$, $\gamma = 0.88$, $\delta = 0.04$, $a = 1.6$ and $b = 40$). (c) Coexisting point attractor at the origin (parameter values are the same as (b)).

$$L_2 \frac{dI}{dt} = -v_r(I) - v_D(i + I)$$
$$C\frac{dv}{dt} = i. \tag{4}$$

By changing the variables and parameters,

$$i = \sqrt{\frac{C}{L_1}} Vx, \quad I = \sqrt{\frac{C}{L_1}} Vy, \quad v = Vz,$$
$$t = \sqrt{L_1 C}\tau, \quad \text{"}\cdot\text{"} = \frac{d}{d\tau},$$
$$\beta = \frac{L_1}{L_2}, \quad \gamma = r\sqrt{\frac{C}{L_1}}, \quad \delta = R\sqrt{\frac{C}{L_1}},$$
$$a = \frac{R_d r_d}{2R_d + r_d}\sqrt{\frac{C}{L_1}}, \quad b = \frac{r_d}{2}\sqrt{\frac{C}{L_1}}, \tag{5}$$

(4) is normalized as

$$\begin{aligned} \dot{x} &= -z - \delta x - f_D(x + y) \\ \dot{y} &= -\beta f_r(y) - \beta f_D(x + y) \\ \dot{z} &= x \end{aligned} \tag{6}$$

where the functions f_r and f_D corresponds to v_r and v_D, respectively, and are represented as follows.

$$f_r(x) = \begin{cases} 1 - \gamma x & \cdots \quad (x > 1/a) \\ (a - \gamma)x & \cdots \quad (|x| \leqq 1/a) \\ -1 - \gamma x & \cdots \quad (x < -1/a), \end{cases} \tag{7}$$

$$f_D(x) = \begin{cases} 3 & \cdots \quad (x > 1/b) \\ 3bx & \cdots \quad (|x| \leqq 1/b) \\ -3 & \cdots \quad (x < -1/b). \end{cases} \tag{8}$$

By circuit experiments and computer calculations, we confirmed that this circuit have the following properties. The origin is always stable and it is surrounded by an unstable limit cycle. Out of the unstable limit cycle, there exists the other stable attractor. As a parameter γ (r) increases continuously, this attractor bifurcates from one-periodic limit cycle to chaos via period-doubling bifurcations.

Figure 5 (a) shows a typical example of chaotic attractors obtained from the circuit in Fig. 2. Figure 5 (b) is the corresponding computer calculated result. Further, Fig. 5 (c) shows that the point attractor at the origin coexist with the chaotic attractor in Fig. 5 (b).

3. Two Coupled Chaotic Oscillators

The main object of this paper is to investigate two chaotic oscillators with hard nonlinearities coupled by a capacitor. The coupled model is shown in Fig. 6. This circuit is considered to be a chaotic circuit version of the coupled oscillators in Fig. 1 (b).

The equation governing the circuit in Fig. 6 is described as follows:

$$\begin{aligned} \dot{x_k} &= -z_k - \alpha(z_1 + z_2) - \delta x_k - f_D(x_k + y_k) \\ \dot{y_k} &= -\beta f_r(y_k) - \beta f_D(x_k + y_k) \\ \dot{z_k} &= x_k \\ &\quad (k = 1, 2) \end{aligned} \tag{9}$$

where $\alpha = C/C_0$ is a new parameter corresponding to the coupling. This circuit equation includes four three-segment piecewise-linear functions. Namely, it is 81-regions piecewise-linear ordinary differential equation.

In the following circuit experiments, the parameter values are fixed as L_1 = 100 mH, L_2 = 200 mH, $C = 0.068\,\mu$F, $R = 50\,\Omega$ and $R_d = 2\,\mathrm{k}\Omega$ and only C_0 and r are varied as a control parameter. While for the computer calculations, the parameter values are fixed as $\beta = 0.5$, $\delta = 0.04$, $a = 1.6$ and $b = 40$. and α and γ are varied. Further, (9) is integrated by using the Runge-Kutta method with step size $h = 0.005$.

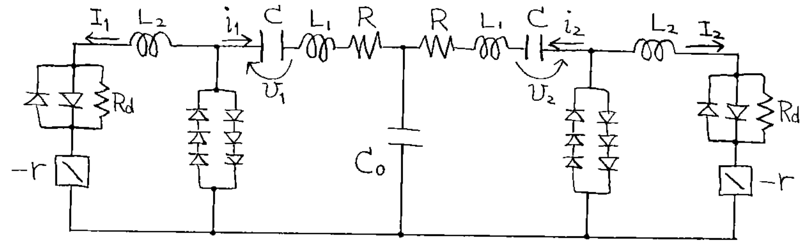

Fig. 6 Two chaotic circuits with hard nonlinearities coupled by a capacitor.

4. Multimode Chaos

From the coupled circuit in Fig. 6, we observed that four different oscillation states coexist; zero, in-phase single-mode, anti-phase single-mode and double-mode.

Zero means that both of two oscillators are not excited. This state is always stable and relatively small initial values are attracted to this state. In the following we do not show any figures of this state, because it is trivial.

In-phase single-mode means that the two oscillators are synchronized at the in-phase. When r increases, the attractor on $i_1 - i_2$ plane is inflated as shown in Fig. 7 (1). However, this state is always one-periodic for the parameter values we treated.

Anti-phase single-mode means that the two oscillators are synchronized with π phase difference. When r increases, this oscillation mode bifurcates from one-periodic state to chaotic state while holding synchronization via the following route. 1-period with symmetry (Fig. 7 (2a)) $\longrightarrow$ 1-period with asymmetry (Fig. 7 (2b)) $\longrightarrow$ (period-doubling bifurcation) $\longrightarrow$ 2-period with asymmetry $\longrightarrow$ 2^n period with asymmetry $\longrightarrow$ chaos with asymmetry (Fig. 7 (2c)) $\longrightarrow$ chaos with symmetry (Fig. 7 (2d)). The observed states of this anti-phase single-mode are similar to those of uncoupled chaotic oscillator with a slight parameter shift. Namely, for the parameter value in Fig. 7 (a), the uncoupled chaotic oscillator exhibits one-periodic attractor with symmetry. For Fig. 7 (b), it exhibits one-periodic with asymmetry. Chaos with asymmetry for (c) and chaos with symmetry for (d).

Double-mode means that above two single-mode oscillations (namely in-phase and anti-phase) are simultaneously excited. This mode of oscillation changes its character according to the character of contained single-mode oscillations. Namely, if both of in-phase and anti-phase oscillations are periodic, the double-mode is simple quasi-periodic oscillation; two-torus (Figs. 7 (3a) and 7 (3b)). However, if the in-phase oscillation is periodic and the anti-phase oscillation is chaotic, the double-mode oscillation must contain both of periodic and chaotic oscillations. As a result, the oscillation waveform looks like amplitude-modulated waveforms of chaotic signal (Figs. 7 (3c) and 7 (3d)). This phenomenon has not been reported yet and we name this as double-mode chaos.

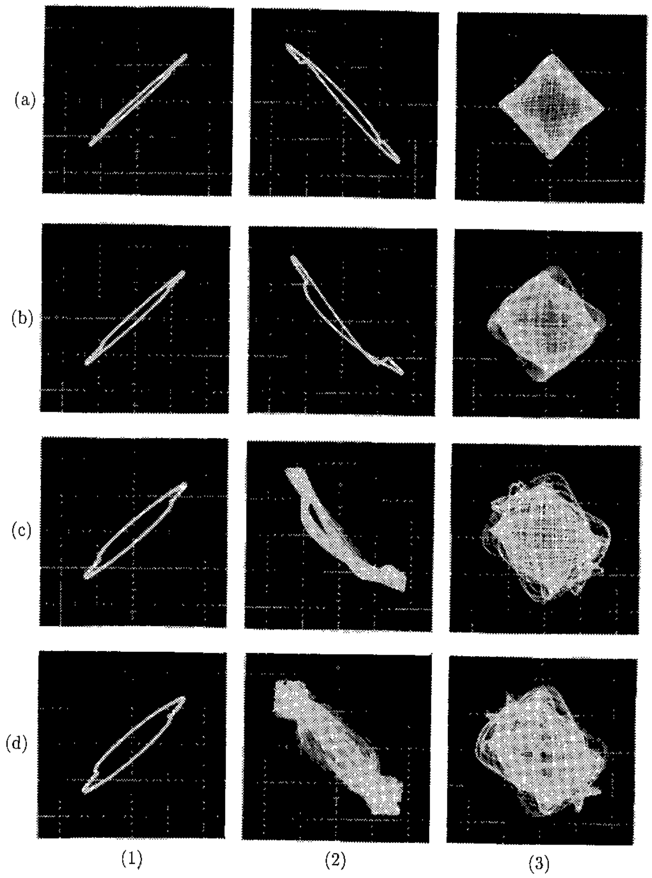

Fig. 7 Bifurcation of three different oscillation modes. Horizontal axis is i_1 (1 mA/div.) and Vertical axis is i_2 (1 mA/div.). $C_0 = 0.34\,\mu$F. (a) $r = 1.20\,\mathrm{k}\Omega$. (b) $r = 1.25\,\mathrm{k}\Omega$. (c) $r = 1.36\,\mathrm{k}\Omega$. (d) $r = 1.41\,\mathrm{k}\Omega$. (1) In-phase single-mode. (2) Anti-phase single-mode. (3) Double-mode.

Figure 8 shows time waveforms and attractors onto $i_1 - v_1$ plane of three different oscillation modes (parameter values are the same as Fig. 7 (d)). We can see that there is difference between frequencies of two single-modes, namely the frequency of in-phase single-mode is higher than that of anti-phase single-mode. Though we cannot give physical explanation, we consider that this difference of frequency is deeply related with the observed bifurcation phenomena of each synchronization mode. Further, we can confirm that the envelopes

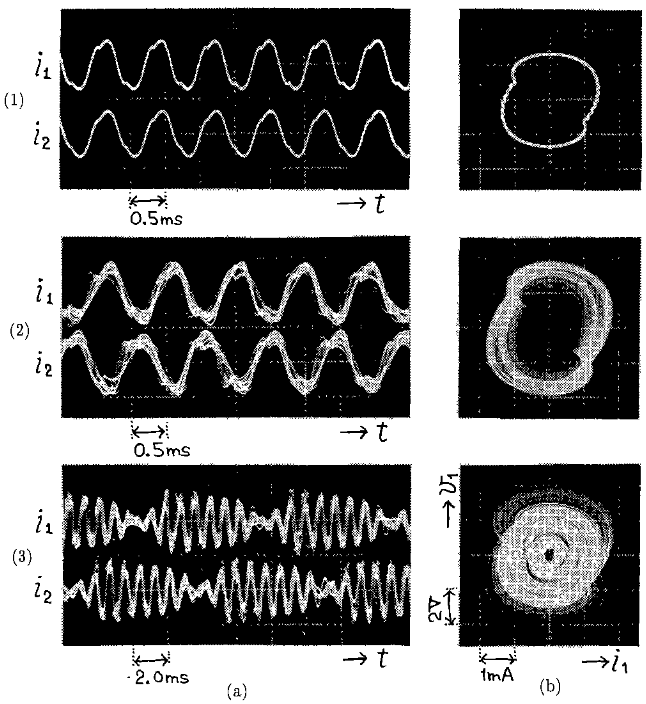

Fig. 8 (a) Time waveforms and (b) attractors observed from a subcircuit. $C_0 = 0.34\,\mu$F and $r = 1.41\,\mathrm{k\Omega}$. (1) In-phase single-mode. (2) Anti-phase single-mode. (3) Double-mode.

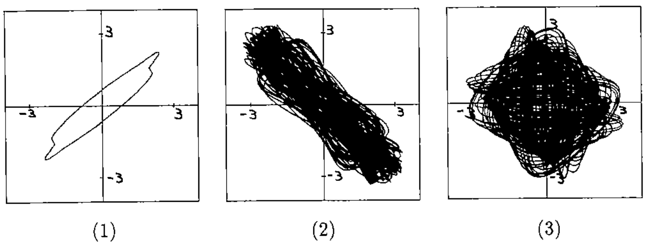

Fig. 9 Computer calculated results corresponding to Fig. 7 (d). Horizontal axis is x_1 and Vertical axis is x_2. $\alpha = 0.25$ and $\gamma = 0.80$.

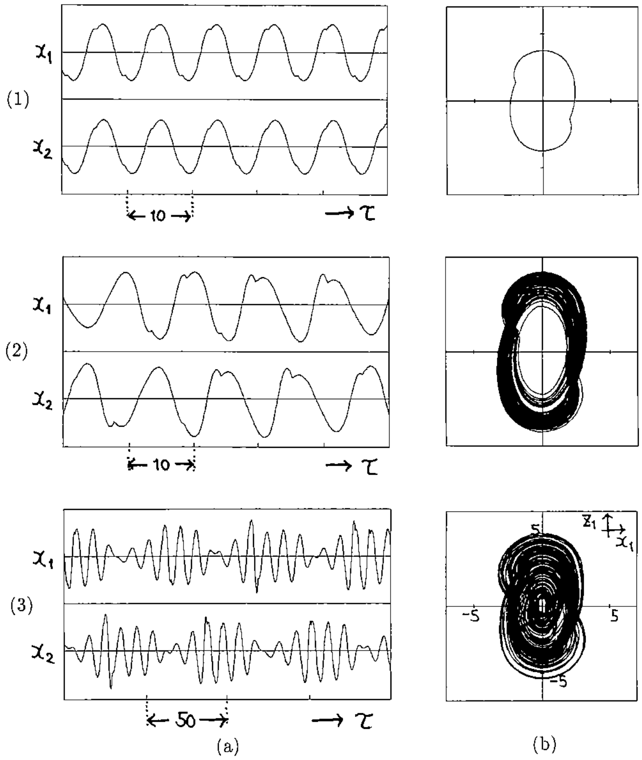

Fig. 10 Computer calculated results corresponding to Fig. 8. $\alpha = 0.25$ and $\gamma = 0.80$.

of double-mode chaos are almost synchronized with π phase difference. This phenomenon is chaotic version of the results in Ref. [4] and extension to large number of oscillators is one of interesting future problems on multimode chaos. Figures 9 and 10 are computer calculated results corresponding to Figs. 7 and 8, respectively. Circuit experimental results and computer calculated results are agree well qualitatively.

Finally, the beat frequency of double-mode chaos can be changed by varying the value of the coupling capacitor C_0 (or α). Figures 11 and 12 show the results obtained by using another values of C_0 (or α).

5. Concluding Remarks

In this study, we have investigated multimode chaos observed from two coupled chaotic oscillators with hard nonlinearities. We confirmed that double-mode chaos

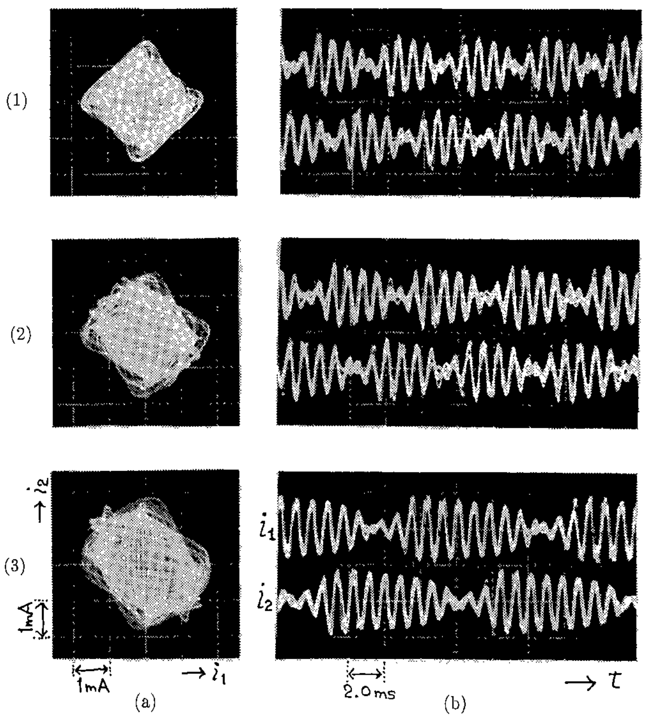

Fig. 11 Change of beat frequency of double-mode chaos. (1) $C_0 = 0.24\,\mu$F and $r = 1.34\,\mathrm{k\Omega}$. (2) $C_0 = 0.27\,\mu$F and $r = 1.38\,\mathrm{k\Omega}$. (3) $C_0 = 0.41\,\mu$F and $r = 1.38\,\mathrm{k\Omega}$.

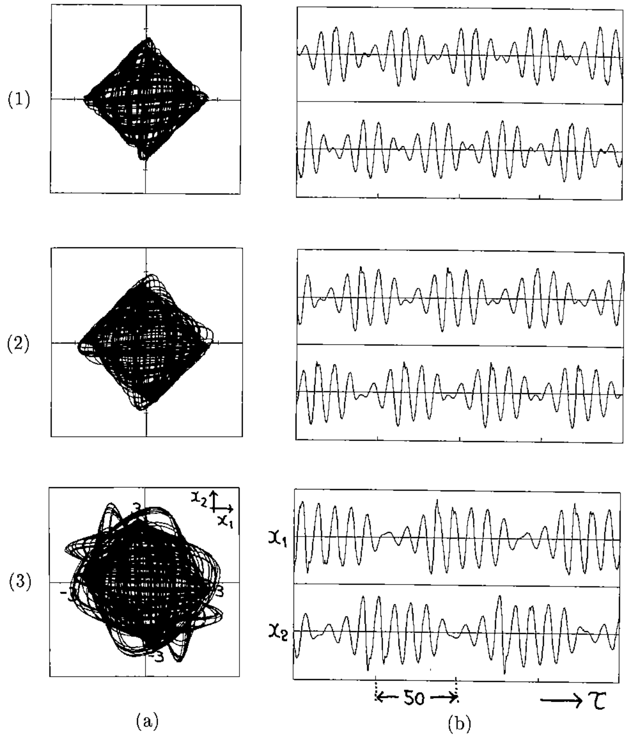

Fig. 12 Computer calculated results corresponding to Fig. 11. (1) $\alpha = 0.330$ and $\gamma = 0.740$. (2) $\alpha = 0.275$ and $\gamma = 0.755$. (3) $\alpha = 0.200$ and $\gamma = 0.800$.

coexisted with two single-modes and zero by circuit experiments and computer calculations.

We consider that there remain many interesting problems on multimode chaos. In the present circuit model, in-phase single-mode was confirmed to be always one-periodic. If in-phase is chaotic as well as anti-phase, double-mode oscillation must contain two different kinds of chaos. Although we tried to search parameters for which such a phenomenon are observed, we could not find. Another chaotic circuits may exhibit such oscillation. Further, extension to large number of oscillators is expected to a triple-mode chaos generator and is also attractive problem.

Acknowledgment

The authors would like to thank Prof. Shinsaku Mori of Keio University for their valuable comments and encouragement.

References

[1] T. Endo and S. Mori, "Mode analysis of a ring of a large number of mutually coupled van der pol oscillators," IEEE Trans. Circuits Syst., vol.25, no.1, pp.7–18, Jan. 1978.

[2] S.P. Datardina and D.A. Linkens, "Multimode oscillations in mutually coupled van der pol type oscillators with fifth-power nonlinear characteristics," IEEE Trans. Circuits Syst., vol.25, no.5, pp.308–315, May 1978.

[3] T. Endo and T. Ohta, "Multimode oscillations in a coupled oscillator system with fifth-power nonlinear characteristics," IEEE Trans. Circuits Syst., vol.27, no.4, pp.277–283, April 1980.

[4] T. Yoshinaga and H. Kawakami, "Synchronized quasi-periodic oscillations in a ring of coupled oscillators with hard characteristics," Trans. IEICE, vol.J75-A, no.12, pp.1811–1818, Dec. 1992.

[5] C. Hayashi, "Nonlinear Oscillations in Physical Systems," Princeton Univ. Press, p.367, 1984.

[6] V.I. Arnold, "Geometrical methods in the theory of ordinary differential equations," Springer-Verlag, pp.270–272, 1988.

[7] For example, the papers in the special session "Advances in theory and applications of chaotic circuits" Proc. of ECCTD'93 and the special session "Synchronization and control of chaotic systems I, II, III and IV" Proc. of ISCAS'95.

[8] Y. Nishio and S. Mori, "Synchronization in mutually coupled chaotic circuits," IEICE Technical Report, NLP92–100, 1993.

[9] Y. Nishio and A. Ushida, "On a ring of chaotic circuits coupled by inductors," IEICE Trans. Fundamentals, vol.E78-A, no.5, pp.608–617, May 1995.

[10] Y. Nishio and A. Ushida, "Spatiotemporal chaos in four chaotic circuits coupled by one resistor," IEICE Technical Report, NLP94–77, 1994.

[11] N. Inaba and S. Mori, "Chaotic phenomena in circuits with a linear negative resistance and an ideal diode," Proc. 31st Midwest Symposium on Circuits Syst., pp.211–214, Aug. 1988.

Yoshifumi Nishio received the B.E., M.E., and Ph.D. degrees in electrical engineering from Keio University, Yokohama Japan, in 1988, 1990, and 1993, respectively. In 1993, he joined the Department of Electrical and Electronic Engineering at Tokushima University, Tokushima Japan, where he is currently an Assistant Professor. His research interests are in chaos and synchronization phenomena in nonlinear circuits. Dr. Nishio is a member of the IEEE.

Akio Ushida received the B.E. and M.E. degrees in electrical engineering from Tokushima University in 1961 and 1966, respectively, and the Ph.D. degree in electrical engineering from University of Osaka Prefecture in 1974. He was an associate professor from 1973 to 1980 at Tokushima University. Since 1980 he has been a Professor in the Department of Electrical Engineering at the university. From 1974 to 1975 he spent one year as a visiting scholar at the Department of Electrical Engineering and Computer Sciences at the University of California, Berkeley. His current research interests include numerical methods and computer-aided analysis of nonlinear system. Dr. Ushida is a member of the IEEE.