PAPER

Synchronization Phenomena in Oscillators Coupled by One Resistor

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SUMMARY There have been many investigations of mutual synchronization of oscillators. In this article, N oscillators with the same natural frequencies mutually coupled by one resistor are analyzed. In this system, various synchronization phenomena can be observed because the system tends to minimize the current through the coupling resistor. When the nonlinear characteristics are third-power, we can observe N-phase oscillation, and this system can take $(N-1)!$ phase states. When the nonlinear characteristics are fifth-power, we can observe $(N-1),(N-2)\cdots 3$ and 2-phase oscillations as well as N-phase oscillations and we can get much more phase states from this system than that of the system with third-power nonlinear characteristics. Because of their coupling structure and huge number of steady states of the system, our system would be a structural element of cellular neural networks. In this study, it is confirmed that our systems can stably take huge number of phase states by theoretical analysis, computer calculations and circuit experiments.
key words: *coupled oscillators, N-phase oscillation, phase states, coupling resistor*

1. Introduction

There have been many investigations of the mutual synchronization of oscillators ([1]–[6] and therein). Endo et al. have analyzed a large number of coupled van der Pol oscillators [1]–[3]. Kimura et al. have confirmed that two oscillators coupled by one resistor are synchronized at opposite phase [4] and that three oscillators coupled by one resistor are synchronized at 3-phase [5]. Further, there have been several investigations of the mutual synchronization of oscillators with fifth-power nonlinear characteristics ([7]–[9]). Endo et al. investigated the multimode oscillations in ladder oscillators with fifth-power nonlinear characteristics [7]. Datardina et al. investigated the multimode oscillations in toe coupled oscillators [8]. Yoshinaga et al. reported the synchronized quasi-periodic oscillations in a ring of coupled oscillators with fifth-power nonlinear characteristics [9].

In Ref. [6], we have reported the synchronization phenomena in N van der Pol oscillators with the same natural frequency mutually coupled by one resistor. In the system various synchronization phenomena can be stably observed, because the system tends to minimize the current through the coupling resistor. Especially, we have confirmed that N-phase oscillations ($N = 2 \sim 13$) can be stably excited when the nonlinearity is strong and in this case there are $(N-1)!$ phase states. This means that when $N = 13$, our system can take 479,001,600 steady states. In Ref. [6], it was confirmed by only computer calculations and circuit experiments.

Further, we have investigated N oscillators with fifth-power nonlinear characteristics coupled by one resistor in Ref. [10]. Because such an oscillator exhibits hard oscillation, we can keep oscillations arbitrary number of oscillations to be stationary. Therefore, we can get n-phase oscillations ($n = 2, 3, \cdots, N$) when N oscillators with fifth-power nonlinearity coupled. As a result, the number of the phase states becomes larger than that of the system with third-power nonlinear characteristics. When $N = 13$, it is considered that we can get 792,712,283 phase states. Because of the coupling structure and extremely large number of phase states of the system, the system would be structural element of the cellular neural network [11] or may be used as an extremely large memory.

In this article, we analyze these two systems and confirm that the system with fifth-power nonlinear characteristics can take larger phase states than that of the system with third-power nonlinear characteristics by computer calculations and circuit experiments.

This paper consists of five sections. Section 1 is the introduction. In Sect. 2, we show the circuit model, and give the circuit equation. In Sect. 3, we show the synchronization phenomena in oscillators with third-power nonlinear characteristics. In Sect. 4, we analyze the synchronization phenomena in oscillators with fifth-power nonlinear characteristics theoretically, and confirm the phenomena by both circuit experiment and computer calculation. Section 5 is the Conclusion.

2. Circuit Model

The circuit model is shown in Fig. 1. The circuit equations are described as follows,

$$C\frac{dv_k}{dt} = -i_k - i_r(v_k)$$

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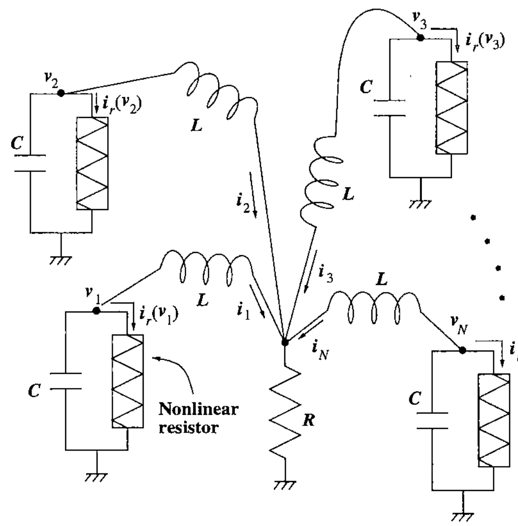


Fig. 1 Circuit model.

$$L\frac{di_k}{dt} = v_k - R\sum_{j=1}^{N} i_j \tag{1}$$
$$(k = 1, 2, \cdots, N)$$

where $i_r(v_k)$ indicates $v - i$ characteristics of nonlinear resistor. In this system, it is approximated by the following function.

$$i_r(v_k) = \begin{cases} -g_1 v_k + g_3 v_k^3 \quad (g_1, g_3 > 0) \\ \text{(third-power} \\ \quad \text{nonlinear characteristics)} \\ g_1 v_k - g_3 v_k^3 + g_5 v_k^5 \quad (g_1, g_3, g_5 > 0) \\ \text{(fifth-power} \\ \quad \text{nonlinear characteristics)} \end{cases} \tag{2}$$

When the nonlinear characteristics are third-power, by changing the variables,

$$t = \sqrt{LC}\tau,$$
$$v_k = \sqrt{\frac{g_1}{3g_3}}x_k, \qquad i_k = \sqrt{\frac{Cg_1}{3Lg_3}}y_k,$$
$$\alpha = R\sqrt{\frac{C}{L}}, \qquad \varepsilon = g_1\sqrt{\frac{L}{C}} \tag{3}$$

Equation (1) is normalized as

$$\dot{x}_k = -y_k + \varepsilon\left(x_k - \frac{x_k^3}{3}\right)$$
$$\dot{y}_k = x_k - \alpha\sum_{j=1}^{N} y_j \tag{4}$$
$$(k = 1, 2, \cdots, N)$$

where $\dot{x} = dx/d\tau$. In Eq. (4), α is the coupling factor and ε is the strength of nonlinearity. When $\alpha, \varepsilon \ll 1$, the oscillation of each oscillators can be regarded as almost purely sinusoidal, and Eq. (4) can written as follows.

$$\ddot{x}_k + x_k = \varepsilon(1 - x_k^2)\dot{x}_k - \alpha\sum_{j=1}^{N}\dot{x}_j \tag{5}$$

In this case, the term $\alpha\varepsilon\sum_{j=1}^{N}(x_j - x_j^3/3)$ can be omitted because it is much smaller than the other terms.

When the nonlinear characteristics are fifth-power, in Eq. (1), by changing variables,

$$t = \sqrt{LC}\tau,$$
$$v_k = \sqrt[4]{\frac{g_1}{5g_5}}x_k, i_k = \sqrt{\frac{C}{L}}\sqrt[4]{\frac{g_1}{5g_5}}y_k,$$
$$\alpha = R\sqrt{\frac{C}{L}}, \beta = \frac{3g_3}{g_1}\sqrt{\frac{g_1}{5g_5}}, \varepsilon = g_1\sqrt{\frac{L}{C}}, \tag{6}$$

Equation (1) is normalized as

$$\dot{x}_k = -y_k - \varepsilon\left(x_k - \frac{1}{3}\beta x_k^3 + \frac{1}{5}x_k^5\right)$$
$$\dot{y}_k = x_k - \alpha\sum_{j=1}^{N} y_j \tag{7}$$
$$(k = 1, 2, \cdots, N)$$

where α is the coupling factor and ε is the strength of nonlinearity. The amplitudes of oscillators depend on β. When $\alpha, \varepsilon \ll 1$, Eq. (7) is written as follows.

$$\ddot{x}_k + x_k = -\varepsilon\left(1 - \beta x_k^2 + x_k^4\right)\dot{x}_k - \alpha\sum_{j=1}^{N}\dot{x}_j \tag{8}$$

In this case, the term $-\alpha\varepsilon\sum_{j=1}^{N}(x_j - \beta x_j^3/3 + x_j^5/5)$ can be omitted just as the case of the third-power nonlinear characteristics.

3. Synchronization Phenomena in the System with Third-Power Nonlinear Characteristics

3.1 Stability of N-Phase Oscillation

In Ref. [4], Kimura et al. have shown the opposite phase oscillation can be seen in two oscillators coupled by one resistor and in Ref. [5], they have shown 3-phase oscillation can be seen in 3 oscillators coupled by one resistor. They have also analyzed these phenomena theoretically by using averaging method. Moreover, we have reported that N-phase oscillations can be stably excited in N oscillators coupled by one resistor in the case of strong nonlinearity [6].

In such systems, to minimize the loss by coupling resistor R, the current through R should be minimized. So the sum of the output voltage $v_1 \sim v_N$ should be 0. When N is a prime number, N-phase oscillations are stably excited. However, when N is not a prime number, we can consider that N-phase oscillation is not excited, because N-phase oscillation does not need to be excited. For example, if $N = 4$, we can see v_1 and v_2 are synchronized at opposite phase and that v_3 and

v_4 are synchronized at opposite phase while the phase between v_1 and v_3 is independent.

We can analyze these phenomena by using averaging method when the nonlinearity is weak (i.e. ε is small). In such cases, we can assume that the waveforms are nearly sinusoidal and that

$$x_k = a_k \cos(\tau - \theta_k).$$

When we assume that the amplitude a_k and the phase θ_k are varying slowly, a_k and θ_k can be described as follows by using averaging method to Eq. (5),

$$\dot{a}_k = \frac{\varepsilon}{2}\left(1 - \frac{1}{4}a_k^2\right)a_k - \frac{\alpha}{2}\sum_{j=1}^{N} a_j \cos(\theta_k - \theta_j) \quad (9)$$

$$\dot{\theta}_k = \frac{\alpha}{2a_k}\sum_{j=1}^{N} a_j \sin(\theta_k - \theta_j) \quad (10)$$

By equating Eqs. (9) and (10) to 0, we can obtain the steady states of a_k and θ_k.

In Refs. [4] and [5], it is theoretically confirmed that the opposite phase and 3-phase oscillation is stably excited. When more than 4 oscillators are coupled, we can analyze the phenomena by the same way. However, because the number of equations become larger, we have to get the eigenvalue of Jacobian by numerical calculation. By this way, we can confirm that N-phase ($N = 4, 5, \cdots$) oscillation cannot be stably excited when the nonlinearity is weak. So we can see N-phase ($N = 4, 5, \cdots$) oscillation only when the nonlinearity is strong as shown in Ref. [6].

3.2 Numerical Analysis and Circuit Experiments

To confirm the synchronization phenomena in the system with third-power nonlinear characteristics, we show the experimental and numerical results. On circuit experiments, the nonlinear resistor is realized as shown in Fig. 2 (a) and $v - i$ characteristics of the resistor when $r = 150\,\Omega$ is shown in Fig. 2 (b). Note that when r is large, nonlinearity is not so strong.

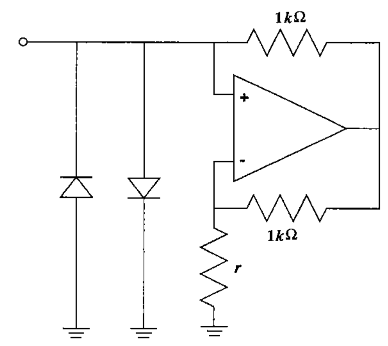
For computer calculation using Runge-Kutta-Gill method, in order to consider the difference between the natural frequency of real oscillators, Eq. (4) is rewritten as follows.

$$\dot{x}_k = -y_k + \varepsilon\left(x_k - \frac{x_k^3}{3}\right)$$
$$\dot{y}_k = (1 + \Delta\omega_k)x_k - \alpha\sum_{j=1}^{N} y_j \quad (11)$$
$$(k = 1, 2, \cdots, N)$$

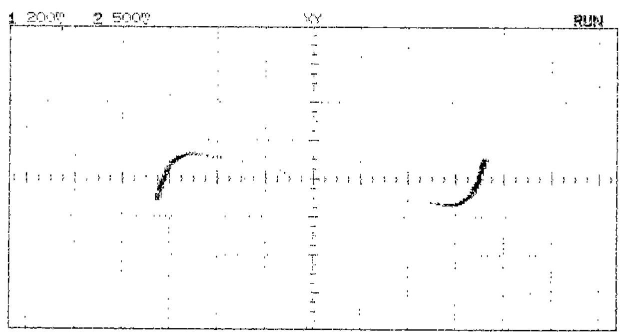
where $\Delta\omega_k$ corresponds to the difference between the natural oscillating frequency of the reference oscillator and those of another oscillator.

We have carried out circuit experiments for the case of $N = 4 \sim 11$. First, we show the results when N is a prime number. When the nonlinearity of each oscillator is weak, N-phase oscillation cannot be excited. Figure 3 shows an example of 5-phase oscillation observed for the case of $N = 5$ by circuit experiments and computer calculation. Similarly, Figs. 4 and 5 show examples for the case of $N = 7, 11$ respectively. From these figures, we can see that 7-phase and 11-phase oscillations are stably excited.

Next, we show the results when N is not a prime

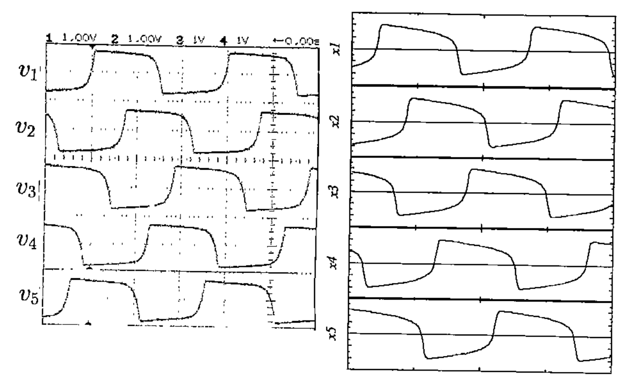


(a) Circuit model of the nonlinear resistor.



(b) $v - i$ characteristics of the nonlinear resistor ($r = 150\Omega$, horizontal scale: 200mV/div., vertical scale: 5mA/div.).

Fig. 2 Realization of nonlinear resistor with third-power nonlinear characteristics.



(a) (b)

Fig. 3 5-phase oscillation obtained from circuit experiment and computer calculation for the case of $N = 5$. (a) Experimental result ($L = 10.0\,$mH, $C = 0.068\,\mu$F, $R = 300\,\Omega$, $r = 150\,\Omega$, horizontal scale: $100\,\mu$s/div., vertical scale: 1.00 V/div.). (b) Numerical result ($\alpha = 2.0$, $\varepsilon = 5.0$, $\max(\Delta\omega_k) = 0.004$).

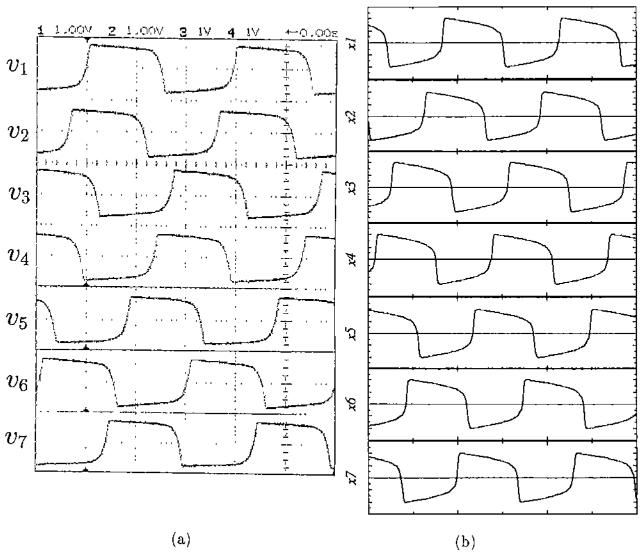


Fig. 4 7-phase oscillation obtained from circuit experiment and computer calculation for the case of $N = 7$. (a) Experimental result (L = 10.0 mH, C = 0.068 μF, R = 300 Ω, r = 150 Ω, horizontal scale: 100 μs/div., vertical scale: 1.00 V/div). (b) Numerical result ($\alpha = 2.0$, $\varepsilon = 6.0$, $\max(\Delta\omega_k) = 0.006$).

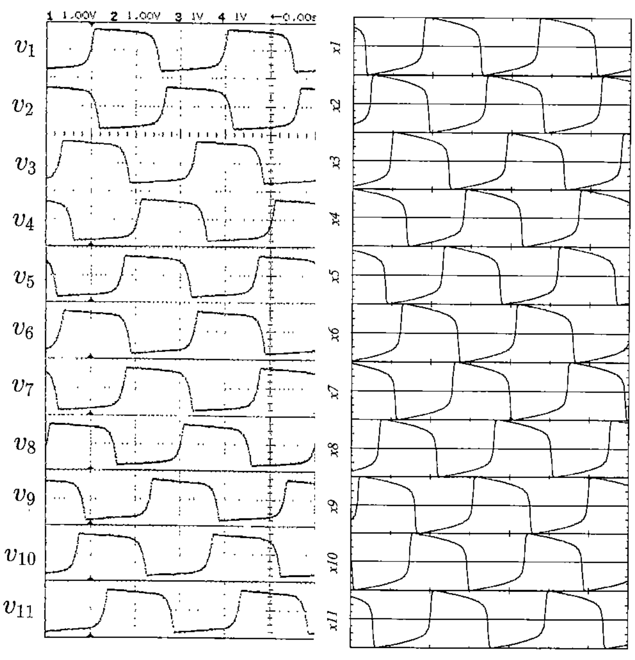


Fig. 5 11-phase oscillation obtained from circuit experiment and computer calculation for the case of $N = 11$. (a) Experimental result (L = 10.0 mH, C = 0.068 μF, R = 500 Ω, r = 150 Ω, horizontal scale: 100 μs/div., vertical scale: 1.00 V/div.). (b) Numerical result ($\alpha = 1.5$, $\varepsilon = 7.0$, $\max(\Delta\omega_k) = 0.001$).

number. Figure 6 shows an example for the case of $N = 4$. When $N = 4$, we cannot see 4-phase oscillation. In this figure, v_1 and v_3 are synchronized at the almost opposite phase and v_2 and v_4 are synchronized at the almost opposite phase. However, the phase differ-

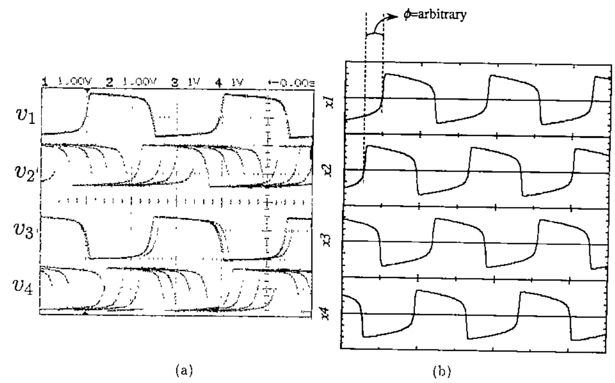


Fig. 6 Result of circuit experiment and computer calculation for the case of $N = 4$. (a) Experimental result (L = 10.0 mH, C = 0.068 μF, R = 300 Ω, r = 150 Ω, horizontal scale: 100 μs/div., vertical scale: 1.00 V/div.). (b) Numerical result ($\alpha = 2.0$, $\varepsilon = 6.5$, $\max(\Delta\omega_k) = 0.003$).

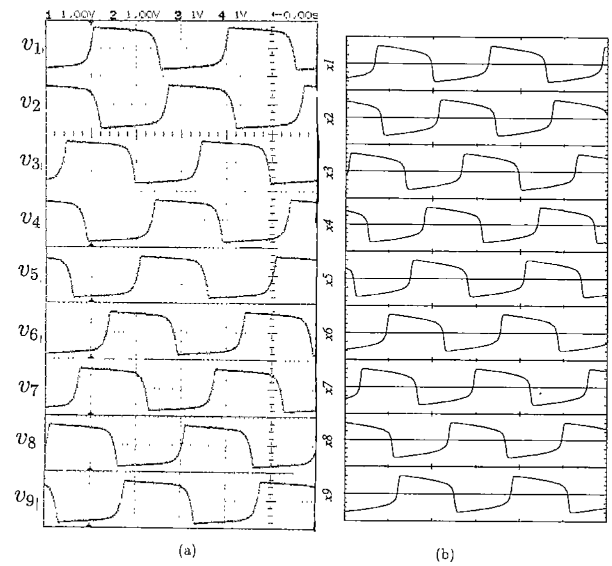


Fig. 7 9-phase oscillation obtained from circuit experiment and computer calculation for the case of $N = 9$. (a) Experimental result (L = 10.0 mH, C = 0.068 μF, R = 300 Ω, r = 150 Ω, horizontal scale: 100 μs/div., vertical scale: 1.00 V/div.). (b) Numerical result ($\alpha = 2.0$, $\varepsilon = 6.0$, $\max(\Delta\omega_k) = 0.004$).

ence between v_1 and v_2 is independent. The combination of the synchronized oscillators seem to be decided by the difference between natural frequencies of real oscillators. Figure 7 shows an example for the case of $N = 9$. When $N = 9$, we considered that we could not see 9-phase oscillation. But from these results, we can see 9-phase oscillation is stably excited by experimental and numerical results.

To confirm the stability of these results, we investigate the stability of fixed points. To find the stability of the fixed points, we calculate the eigenvalues of the Jacobian of Poincaré map. In Table 1, we show the maximum eigenvalues for some parameters of α and ε. When the absolute values of eigenvalues are less than 1,

Table 1 Eigenvalues of the Jacobian of the Poincaré map of the system with third-power nonlinear characteristics.

(a) $N = 5$.

ε	α 1.0	α 2.0
5.0	0.9310294	0.7880829
3.0	0.8891462	0.8994127
1.0	0.9970051	0.9874986

(b) $N = 7$.

ε	α 1.0	α 2.0
7.0	0.8445946	0.8253883
6.0	0.9646250	0.8567104
4.0	0.9025337	0.9251142

(c) $N = 9$.

ε	α 1.0	α 2.0
6.0	0.9569461	0.9687292
5.0	0.9565380	0.9418810

(d) $N = 11$.

ε	α 1.0	α 2.0
7.0	0.9790941	0.9950770
6.0	0.9621364	0.9686279

the fixed points are stable. From these tables, we can confirm that N-phase oscillation can be stably excited when α and ε are large.

4. Synchronization Phenomena in the System with Fifth-Power Nonlinear Characteristics

4.1 Theoretical Analysis

The oscillator with the fifth-power nonlinear characteristics is normalized as following equation.

$$\ddot{x} + x = -\varepsilon(1 - \beta x^2 + x^4)\dot{x} \tag{12}$$

Using averaging method, the amplitude a of x is described as follows.

$$\dot{a} = -\frac{\varepsilon}{2}\left(1 - \frac{\beta}{4}a^2 + \frac{a^4}{8}\right)a \tag{13}$$

Let $A = a^2$,

$$\dot{A} = -\varepsilon\left(1 - \frac{\beta}{4}A + \frac{A^2}{8}\right)A \tag{14}$$

Considering the stability of this equation,

1. When $A(0) > \beta - \sqrt{\beta^2 - 8}$, $\quad A \to \beta + \sqrt{\beta^2 - 8}$
2. When $A(0) < \beta - \sqrt{\beta^2 - 8}$, $\quad A \to 0$

So we can see whether the oscillation is excited or not depends on its initial state. Using these characteristics, it is considered that we can keep arbitrary number of oscillations to be stationary.

Using averaging method, Eq. (8) is described as

$$\dot{a}_k = \frac{\varepsilon}{2}\left\{-1 + \frac{\beta}{4}a_k^2 - \frac{1}{8}a_k^4\right\}a_k - \frac{\alpha}{2}\sum_{j=1}^{N} a_j \cos(\theta_k - \theta_j) \tag{15}$$

$$\dot{\theta}_k = \frac{\alpha}{2a_k}\sum_{j=1}^{N} a_j \sin(\theta_k - \theta_j) \tag{16}$$

By equating Eqs. (15) and (16) to 0, we can obtain the steady states of a_k and θ_k by the same way in Refs. [4] and [5].

When $N = 2$, we can take the following equations from Eq. (16).

$$\dot{\theta}_1 = \frac{\alpha a_2}{2a_1}\sin(\theta_1 - \theta_2) \tag{17}$$

$$\dot{\theta}_2 = \frac{\alpha a_1}{2a_2}\sin(\theta_2 - \theta_1) \tag{18}$$

When we take

$$\theta_1 - \theta_2 = \theta \tag{19}$$

$$\theta_{10} - \theta_{20} = \theta_0 \tag{20}$$

where θ_{10} and θ_{20} are the steady states values of θ_1 and θ_2. Therefore, we can get the conditions as follows,

$$\sin\theta_0 = 0 \tag{21}$$

and

$$\dot{\theta} = \frac{\alpha}{2}\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right)\sin\theta \tag{22}$$

The condition of stability of θ when $\theta = \theta_0$ is described as

$$\left[\frac{\partial}{\partial\theta}\left(\frac{d\theta}{d\tau}\right)\right]_{\theta=\theta_0} = \frac{\alpha}{2}\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right)\cos\theta_0 < 0 \tag{23}$$

From Eqs. (21) and (23),

$$\cos\theta_0 = -1 \tag{24}$$

So the steady states of the phase difference θ_0 are

$$\theta_0 = (2n+1)\pi \quad (n = \pm 1, \pm 2, \cdots). \tag{25}$$

Then we can see x_1 and x_2 synchronize at the opposite phase.

Substituting this relation into Eq. (15),

$$\frac{\varepsilon}{2}\left\{-1 + \frac{\beta}{4}a_{10}^2 - \frac{1}{8}a_{10}^4\right\}a_{10} - \frac{\alpha}{2}(a_{10} - a_{20}) = 0 \tag{26}$$

$$\frac{\varepsilon}{2}\left\{-1 + \frac{\beta}{4}a_{20}^2 - \frac{1}{8}a_{20}^4\right\}a_{20} + \frac{\alpha}{2}(a_{10} - a_{20}) = 0. \tag{27}$$

From these equations,

$$(a_{10}-a_{20})\left[\frac{\varepsilon}{2}\left\{1-\frac{\varepsilon}{4}(a_{10}^2+a_{10}a_{20}+a_{20}^2)+\frac{1}{8}(a_{10}^4+a_{10}^3a_{20}+a_{10}^2a_{20}^2+a_{10}a_{20}^3+a_{20}^4)\right\}+\alpha\right]=0. \tag{28}$$

So we can get the relation

$$a_{10}=a_{20} \tag{29}$$

From Eq. (15)

$$a_{10}=a_{20}=\begin{cases}0, & \text{for } A(0)<\beta-\sqrt{\beta^2-8}\\ \sqrt{\beta+\sqrt{\beta^2-8}}, & \text{for } A(0)>\beta-\sqrt{\beta^2-8}\end{cases} \tag{30}$$

Therefore, we can see that x_1 and x_2 synchronize at the opposite phase or that both x_1 and x_2 are 0.

When $N=3$, using averaging method,

$$\dot{a_1}=\frac{\varepsilon}{2}\left\{-\left(1+\frac{\alpha}{\varepsilon}\right)+\frac{\beta}{4}a_1^2-\frac{1}{8}a_1^4\right\}a_1-\frac{\alpha a_2}{2}\cos(\theta_1-\theta_2)-\frac{\alpha a_3}{2}\cos(\theta_3-\theta_1) \tag{31}$$

$$\dot{a_2}=\frac{\varepsilon}{2}\left\{-\left(1+\frac{\alpha}{\varepsilon}\right)+\frac{\beta}{4}a_2^2-\frac{1}{8}a_2^4\right\}a_2-\frac{\alpha a_1}{2}\cos(\theta_1-\theta_2)-\frac{\alpha a_3}{2}\cos(\theta_2-\theta_3) \tag{32}$$

$$\dot{a_3}=\frac{\varepsilon}{2}\left\{-\left(1+\frac{\alpha}{\varepsilon}\right)+\frac{\beta}{4}a_3^2\frac{1}{8}a_3^4\right\}a_3-\frac{\alpha a_1}{2}\cos(\theta_3-\theta_1)-\frac{\alpha a_2}{2}\cos(\theta_2-\theta_3) \tag{33}$$

$$\dot{\theta_1}=\frac{\alpha a_2}{2a_1}\sin(\theta_1-\theta_2)-\frac{\alpha a_3}{2a_1}\sin(\theta_3-\theta_1) \tag{34}$$

$$\dot{\theta_2}=-\frac{\alpha a_1}{2a_2}\sin(\theta_1-\theta_2)+\frac{\alpha a_3}{2a_2}\sin(\theta_2-\theta_3) \tag{35}$$

$$\dot{\theta_3}=\frac{\alpha a_1}{2a_3}\sin(\theta_3-\theta_1)-\frac{\alpha a_2}{2a_3}\sin(\theta_2-\theta_3) \tag{36}$$

Equating Eqs. (34)–(36) to 0, we can get the relations described as below.

$$a_{20}\sin(\theta_{10}-\theta_{20})=a_{30}\sin(\theta_{30}-\theta_{10}) \tag{37}$$

$$a_{10}\sin(\theta_{10}-\theta_{20})=a_{30}\sin(\theta_{20}-\theta_{30}) \tag{38}$$

$$a_{20}\sin(\theta_{20}-\theta_{30})=a_{10}\sin(\theta_{30}-\theta_{10}) \tag{39}$$

Now we take

$$\begin{cases}\theta_1-\theta_2=\theta_{12}\\ \theta_2-\theta_3=\theta_{23}\end{cases}$$

then,

$$\theta_{31}=\theta_3-\theta_1=-(\theta_{12}+\theta_{23}).$$

And we take

$$\begin{cases}\theta_{10}-\theta_{20}=\hat{\theta}_{12}\\ \theta_{20}-\theta_{30}=\hat{\theta}_{23}\\ \theta_{30}-\theta_{10}=\hat{\theta}_{31}\end{cases}$$

From Eqs. (37)–(39), we can get the relations between the phases and the amplitude as follows because $\theta_{12}+\theta_{23}+\theta_{31}=0$.

$$\cos\hat{\theta}_{12}=\frac{1}{2}\left(\frac{a_{30}^2}{a_{10}a_{20}}-\frac{a_{10}}{a_{20}}-\frac{a_{20}}{a_{10}}\right) \tag{40}$$

$$\cos\hat{\theta}_{23}=\frac{1}{2}\left(\frac{a_{10}^2}{a_{20}a_{30}}-\frac{a_{20}}{a_{30}}-\frac{a_{30}}{a_{20}}\right) \tag{41}$$

$$\cos\hat{\theta}_{31}=\frac{1}{2}\left(\frac{a_{20}^2}{a_{10}a_{30}}-\frac{a_{10}}{a_{30}}-\frac{a_{30}}{a_{10}}\right) \tag{42}$$

Substituting Eqs. (40)–(42) into Eq. (15), we get

$$\frac{\varepsilon}{2}\left\{-1+\frac{\beta}{4}a_{10}^2-\frac{1}{8}a_{10}^4\right\}a_{10}=0 \tag{43}$$

$$\frac{\varepsilon}{2}\left\{-1+\frac{\beta}{4}a_{20}^2-\frac{1}{8}a_{20}^4\right\}a_{20}=0 \tag{44}$$

$$\frac{\varepsilon}{2}\left\{-1+\frac{\beta}{4}a_{30}^2-\frac{1}{8}a_{30}^4\right\}a_{30}=0 \tag{45}$$

If all the oscillators are excited,

$$a_{10}=a_{20}=a_{30}. \tag{46}$$

Substituting Eq. (46) into Eqs. (40)–(42),

$$\cos\hat{\theta}_{12}=\cos\hat{\theta}_{23}=\cos\hat{\theta}_{31}=-\frac{1}{2} \tag{47}$$

So the phase difference between each oscillator is considered $\pm\frac{2}{3}\pi$.

Next, we have to confirm the stability of the phase difference. From Eq. (46), Eqs. (34)–(36) are described as follows.

$$\dot{\theta_1}=\frac{\alpha}{2}\{\sin(\theta_1-\theta_2)-\sin(\theta_3-\theta_1)\} \tag{48}$$

$$\dot{\theta_2}=\frac{\alpha}{2}\{\sin(\theta_1-\theta_2)-\sin(\theta_2-\theta_3)\} \tag{49}$$

$$\dot{\theta_3}=\frac{\alpha}{2}\{\sin(\theta_3-\theta_1)-\sin(\theta_2-\theta_3)\} \tag{50}$$

From Eqs. (48)–(50),

$$\begin{aligned}\dot{\theta}_{12}&=\frac{\alpha}{2}\{2\sin\theta_{12}-\sin\theta_{23}+\sin(\theta_{12}+\theta_{23})\}\\&=P(\theta_{12},\theta_{23})\\ \dot{\theta}_{23}&=\frac{\alpha}{2}\{2\sin\theta_{23}-\sin\theta_{12}+\sin(\theta_{12}+\theta_{23})\}\\&=Q(\theta_{12},\theta_{23})\end{aligned} \tag{51}$$

The stability condition of these differential equations is given as

$$\left.\begin{aligned}p&=-\left(\frac{\partial P}{\partial\theta_{12}}+\frac{\partial Q}{\partial\theta_{23}}\right)>0\\ q&=\begin{vmatrix}\frac{\partial P}{\partial\theta_{12}} & \frac{\partial P}{\partial\theta_{23}}\\ \frac{\partial Q}{\partial\theta_{12}} & \frac{\partial Q}{\partial\theta_{23}}\end{vmatrix}>0\end{aligned}\right\} \tag{52}$$

Substituting Eq. (51) into Eq. (52),

$$p = -\frac{\alpha}{2} \cdot 2\{\cos\theta_{12} + \cos\theta_{23} + \cos(\theta_{12} + \theta_{23})\} > 0$$

$$\cos\theta_{12} + \cos\theta_{23}\cos(\theta_{12} + \theta_{23}) < 0 \quad (53)$$

$$q = \frac{\alpha^2}{4} \cdot 3\{\cos\theta_{12}\cos\theta_{23} + (\cos\theta_{12} + \cos\theta_{23})\cos(\theta_{12} + \theta_{23})\} > 0 \quad (54)$$

$$\cos\theta_{12}\cos\theta_{23} + (\cos\theta_{12} + \cos\theta_{23})\cos(\theta_{12} + \theta_{23}) > 0$$

Under the condition Eq. (47), Eqs. (53) and (54) are satisfied.

Next, we consider the case that one oscillator isn't excited, for example, $x_3 = 0$. In this case, from Eq. (40),

$$\cos\hat{\theta}_{12} = \frac{1}{2}\left(-\frac{a_{10}}{a_{20}} - \frac{a_{20}}{a_{10}}\right) \quad (55)$$

while $\cos\hat{\theta}_{23}$ and $\cos\hat{\theta}_{31}$ are infinity. Because the amplitude of x_3 is 0, the phase differences of x_1 and x_2 from x_3 are nonsense. So it is enough that we investigate the phase difference only between x_1 and x_2. The amplitudes of x_1 and x_2 have the following relation.

$$a_{10} = a_{20} \quad (56)$$

So Eq. (55) is written as

$$\cos\hat{\theta}_{12} = -1 \quad (57)$$

Because this relation is the same as $N = 2$, it is considered that x_1 and x_2 synchronize at the opposite phase. It is easy to predict that the synchronization phenomena like this can be seen when x_2 or x_3 is 0. Therefore, in $N = 3$, we can see both the opposite and 3-phase oscillations in the system with fifth-power nonlinear characteristics and we can take that five phase states is stably excited. Moreover, it is stable when $x_1 = x_2 = x_3 = 0$. Therefore, when $N = 3$, the number of the steady states of the system with fifth-power nonlinear characteristics is 5.

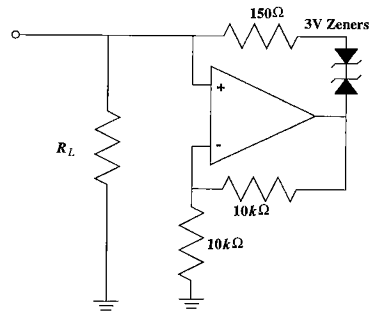
When more than 4 oscillators are coupled, we can analyzed the phenomena by the same way. However, in such cases, N-phase oscillation cannot be excited by the reason which indicated in Sect. 3.1. With strong nonlinearity, we can predict that N-phase oscillation can be stably excited when every oscillator is excited. If n $(n = 1, 2, \cdots, N-2)$ oscillators are not excited, the excited oscillators are not influenced by the stopped oscillators because the current through the coupling resistor is 0, we complain that $(N-n)$-phase oscillation can be excited. So we investigate the stability of the system with strong nonlinearity in Sect. 4.2. If all the patterns are stable in larger N, the number of the steady states of the system with fifth-power nonlinear characteristics P_N is described as follows.

$$P_N = \sum_{n=2}^{N} {}_N C_n \cdot p_n + 1, \quad (58)$$

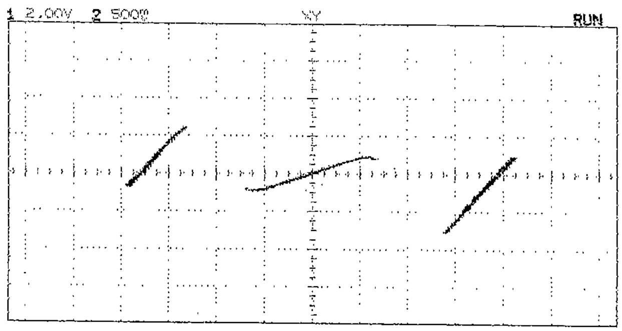
where p_n is the number of the steady states of the system with third-power nonlinear characteristics. Obviously, we can get much more steady states from the system with fifth-power nonlinear characteristics than that of the system with third-power nonlinear characteristics. In Table 2, we show comparison of the number of steady states between the systems with third-power and fifth-power nonlinear characteristics.

Table 2 Comparison of the number of steady states.

N	with third-power nonlinear character	with fifth-power nonlinear character
2	1	2
3	2	6
4	3	18
5	24	70
6	15	280
⋮	⋮	⋮
13	479,001,600	792,712,284



(a) Circuit model of the nonlinear resistor.



(b) $v - i$ characteristics of the nonlinear resistor (R_L = 1.0kΩ, horizontal scale: 2.0V/div., vertical scale: 5.0mA/div.).

Fig. 8 Realization of nonlinear resistor with fifth-power nonlinear characteristics.

4.2 Numerical Analysis and Circuit Experiments

In this section, we show the experimental and numerical results to confirm the synchronization phenomena in oscillators coupled by one resister. In circuit experiment, we use the circuit shown in Fig. 8 (a) as nonlinear resistors. Figure 8 (b) shows $v - i$ characteristics of the nonlinear resistor. Just as the case of third-power non-

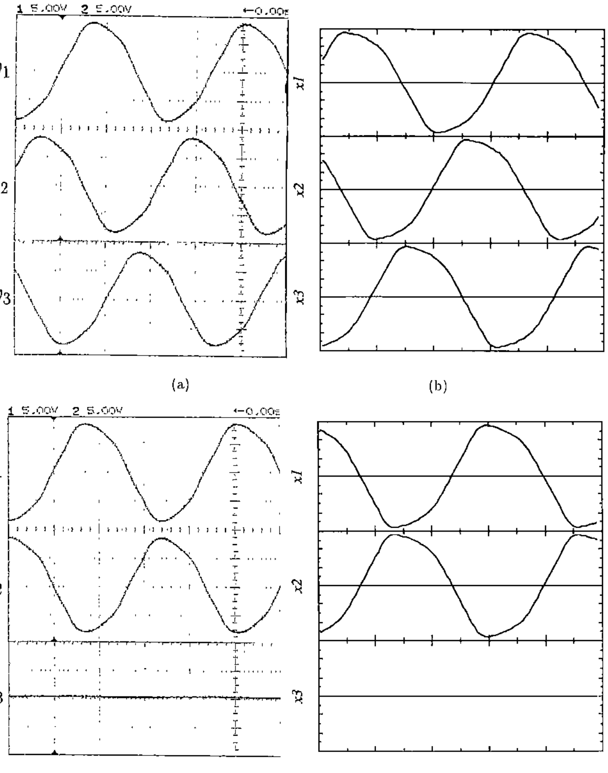


Fig. 9 Experimental and numerical results for the case of $N = 3$. (a) Experimental result when 3 oscillators are excited ($L = 10.0\,\mathrm{mH}$, $C = 0.068\,\mu\mathrm{F}$, $R_L = 1.0\,\mathrm{k\Omega}$, $R = 300\,\Omega$, horizontal scale: $50.0\,\mu$s/div., vertical scale: 5.0 V/div.) (b) Numerical result when 3 oscillators are excited ($\alpha = 4.0$, $\beta = 4.4$, $\varepsilon = 0.2$, $\max(\Delta\omega_k) = 0.002$). (c) Experimental result when 2 oscillators are excited ($L = 10.0\,\mathrm{mH}$, $C = 0.068\,\mu\mathrm{F}$, $R_L = 1.0\,\mathrm{k\Omega}$, $R = 300\,\Omega$, horizontal scale: $50.0\,\mu$s/div., vertical scale: 5.0 V/div.) (d) Numerical result when 2 oscillators are excited ($\alpha = 4.0$, $\beta = 4.4$, $\varepsilon = 0.2$, $\max(\Delta\omega_k) = 0.002$).

linear characteristics, Eq. (7) is rewritten as follows for computer calculation.

$$\begin{aligned}\dot{x}_k &= -y_k - \varepsilon\left(x_k - \frac{1}{3}\beta x_k^3 + \frac{1}{5}x_k^5\right)\\ \dot{y}_k &= (1 + \Delta\omega_k)x_k - \alpha\sum_{j=1}^{N} y_j \qquad (59)\\ (k &= 1, 2, \cdots, N)\end{aligned}$$

When N is larger than five and the nonlinearity is weak, we cannot see N-phase oscillation as the case with the third-power nonlinear characteristics by the reason shown in Sect. 3.1. So we show the experimental and numerical results when nonlinearity is strong. Figures 9 and 10 show the results for the case of $N = 3, 4$, respectively. From Fig. 9, we can see 3-phase oscillation when 3 oscillators are excited and also opposite phase oscillation when one oscillator is stopped. Similarly, from Fig. 10, we can see that v_1 and v_4 synchronize at the opposite phase and v_2 and v_3 synchronize at the opposite phase but the phase difference between v_1 and v_2 is independent. And we can also see both 3-phase (see Figs. 10 (c) and (d)) and opposite phase oscillations. Figure 11 shows the results for the case of $N = 5$. We can see 5-phase oscillation because of strong nonlinearity. We can see also the pairs of the opposite oscillation, 3-phase oscillation, (see Figs. 11 (c) and (d)) and the opposite oscillation. From these results, we can confirm that we can take much more phase states than that of the system with third-power nonlinear characteristics.

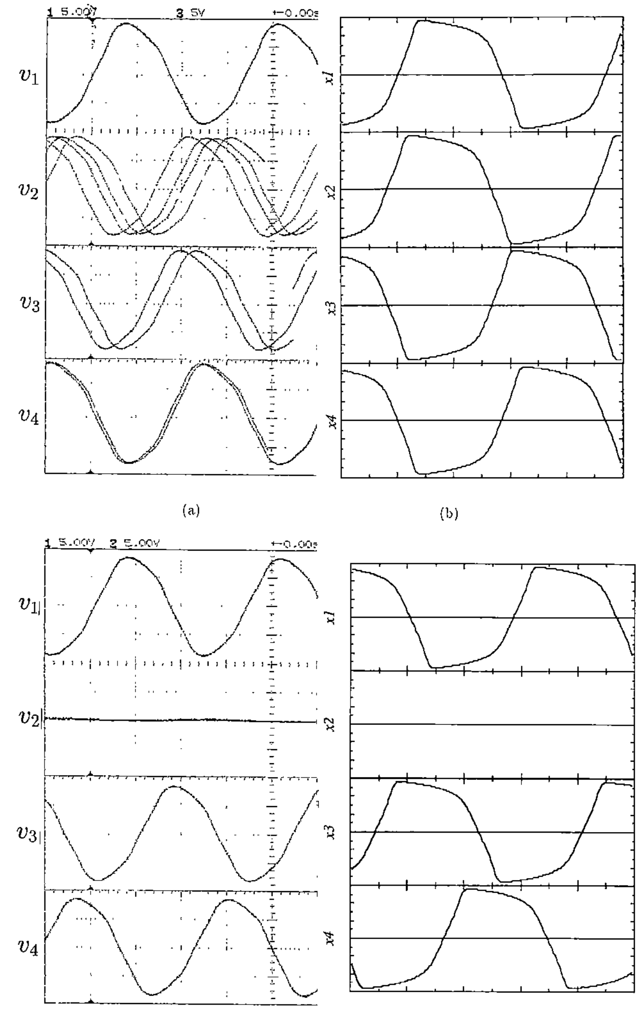


Fig. 10 Experimental and numerical results for the case of $N = 4$. (a) Experimental result when 4 oscillators are excited ($L = 10.0\,\mathrm{mH}$, $C = 0.068\,\mu\mathrm{F}$, $R_L = 1.0\,\mathrm{k\Omega}$, $R = 300\,\Omega$, horizontal scale: $50.0\,\mu$s/div., vertical scale: 5.0 V/div.). (b) Numerical result when 4 oscillators are excited ($\alpha = 3.0$, $\beta = 4.5$, $\varepsilon = 0.5$, $\max(\Delta\omega_k) = 0.003$). (c) Experimental result when 3 oscillators are excited ($L = 10.0\,\mathrm{mH}$, $C = 0.068\,\mu\mathrm{F}$, $R_L = 1.0\,\mathrm{k\Omega}$, $R = 300\,\Omega$, horizontal scale: $50.0\,\mu$s/div., vertical scale: 5.0 V/div.). (d) Numerical result when 3 oscillators are excited ($\alpha = 3.0$, $\beta = 4.5$, $\varepsilon = 0.5$, $\max(\Delta\omega_k) = 0.003$).

To confirm the stability of these results, we investigate the stability of fixed points by calculating the eigen-

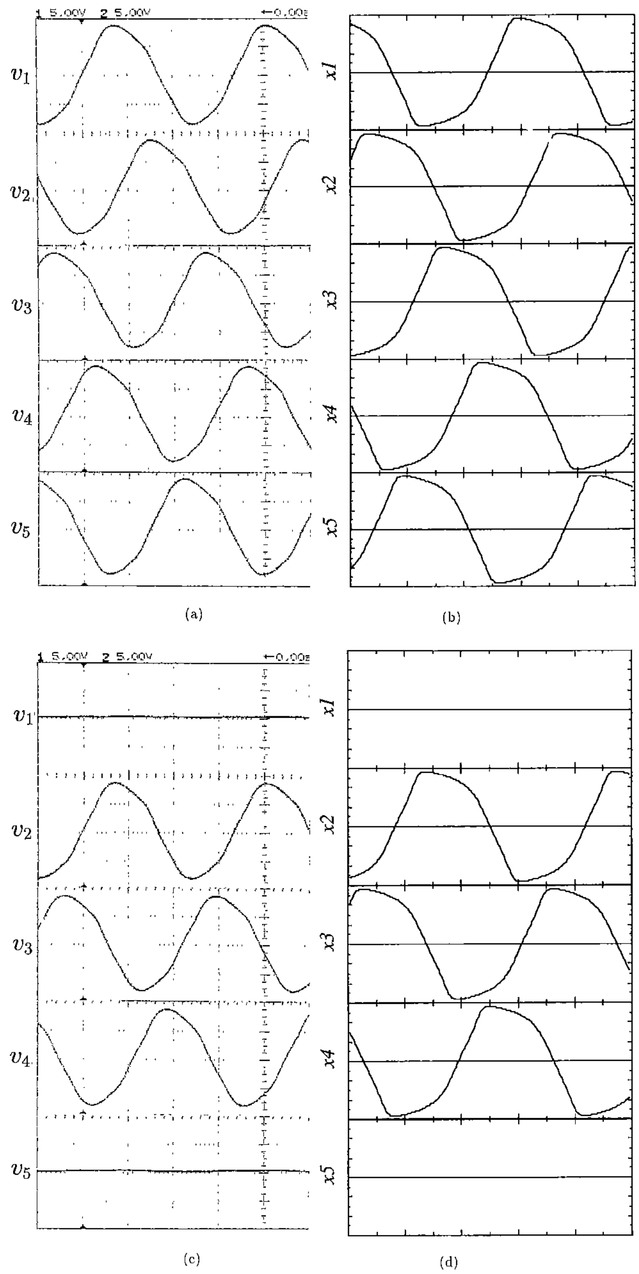


Fig. 11 Experimental and numerical results for the case of $N = 5$. (a) Experimental result when 5 oscillators are excited ($L = 10.0\,\text{mH}$, $C = 0.068\,\mu\text{F}$, $R_L = 1.0\,\text{k}\Omega$, $R = 300\,\Omega$, horizontal scale: $50.0\,\mu$s/div., vertical scale: 5.0 V/div.). (b) Numerical result when 5 oscillators are excited ($\alpha = 4.0$, $\beta = 4.5$, $\varepsilon = 0.3$, $\max(\Delta\omega_k) = 0.004$). (c) Experimental result when 3 oscillators are excited ($L = 10.0\,\text{mH}$, $C = 0.068\,\mu\text{F}$, $R_L = 1.0\,\text{k}\Omega$, $R = 300\,\Omega$, horizontal scale: $50.0\,\mu$s/div., vertical scale: 5.0 V/div.). (d) Numerical result when 3 oscillators are excited ($\alpha = 4.0$, $\beta = 4.5$, $\varepsilon = 0.3$, $\max(\Delta\omega_k) = 0.004$).

values of the Jacobian of Poincaré map. In Table 3, we show the maximum eigenvalues for the synchronization patterns corresponding to the experimental and numerical results. When the absolute values of the eigenvalues are less than 1, the fixed points are stable. From these

Table 3 Eigenvalues of the Jacobian of Poincaré map of the system with fifth-power nonlinear characteristics.

(a) $N = 3$ ($\alpha = 4.0, \beta = 4.4, \varepsilon = 0.2$).

Number of oscillators excited	Eigenvalue
3	0.6619829
2	0.7385893

(b) $N = 4$ ($\alpha = 3.0, \beta = 4.5, \varepsilon = 0.5$).

Number of oscillators excited	Eigenvalue
3	0.5277383
2	0.5814649

(b) $N = 5$ ($\alpha = 4.0, \beta = 4.5, \varepsilon = 0.3$).

Number of oscillators excited	Eigenvalue
5	0.7250972
3	0.8298949
2	0.7720362

tables, we can confirm that not only N-phase oscillation but also $N-1, N-2, \cdots, 3, 2$-phase oscillations can be stably excited when the nonlinear characteristics are fifth-power. Therefore, we can get many phase states as shown in Eq. (58).

5. Conclusion

In this study, we have investigated the synchronization phenomena in oscillators coupled by one resistor with both third and fifth power nonlinear characteristics.

In the system with third-power nonlinear characteristics, we can see N-phase oscillation when N is a prime number. In this case, we can get $(N-1)!$ phase states from the system. Moreover, in the system with fifth-power nonlinear characteristics, we can see $N, (N-1) \cdots 3$ and 2-phase oscillations and can get much more phase states than that of the system with third-power nonlinear characteristics.

Because of the coupling structure and extremely large number of steady states of the systems, they would be utilized as a structural element of cellular neural network or an extremely large memory.

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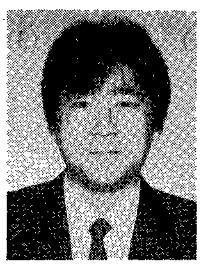

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