Time-Evolving Bifurcation Phenomena in a Chaotic Circuit with a Memristor

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Abstract—**In this study, we investigated the dynamic behavior of a four-dimensional chaotic circuit incorporating a memristor. Numerical simulations demonstrated that the circuit exhibited temporal switching between periodic and chaotic oscillations. This switching phenomenon suggests that the bifurcation parameters may change continuously depending on the dynamics of the internal state of the memristor, providing a new perspective on the role of the memristor in the chaotic circuit.**

I. Introduction

Memristor is a two-terminal nonlinear device whose resistance varies with the applied electric charge or magnetic flux. The first concept was proposed by Leon O. Chua in 1971, and a physical realization was developed by Hewlett-Packard (HP) Labs in 2008. Due to its unique nonlinear resistance-changing properties, the memristor has attracted significant attention across various fields, particularly in next-generation memory technologies [1], soft computing circuits [2], modeling of dynamic behaviors in biological systems [3], and chaotic systems [4].

Chaotic systems incorporating memristors have been reported to exhibit complex dynamical behaviors, such as hyperchaos and the emergence of equilibrium states that differ from those observed in conventional chaotic systems. For this reason, detailed analysis of memristor-based chaotic systems is important not only for its academic significance in offering new analytical perspectives on nonlinear dynamics, but also for its potential in designing new dynamical systems that effectively utilize such complex behaviors. However, it is generally known that systems exhibiting a wide range of nonlinear behaviors tend to be high-dimensional and difficult to analyze. Furthermore, although several studies have reported on the role of memristors in the observed phenomena, only a limited number have clearly elucidated their specific influence within the system.

Therefore, in this study, we focus on a four-dimensional chaotic circuit in which memristors are introduced into a simple chaotic oscillator that uses two diodes as nonlinear elements, originally proposed by Nishio et al. [5]. The base circuit is a self-excited oscillator, and previous studies have shown that it exhibits a variety of bifurcation phenomena depending on the control parameters. The aim of this study is to show the effect of memristors on chaotic circuits through investigation of oscillations and bifurcation in this circuit.

II. Circuit Model

In this study, we use the HP memristor model. This memristor exhibits a pinched hysteresis loop in its v–i characteristics. Figure 1 shows the v–i response when a sinusoidal voltage is applied. The resistance of the memristor, referred to as the memristance $M(q)$, is defined as a function of the charge $q(t)$ that has passed through the device, as shown in Eq. (1).

$$M(q) = \mu_v \frac{{R^2}_{\mathrm{on}}}{D^2} q(t) + R_{\mathrm{off}} \left(1 - \mu_v \frac{R_{\mathrm{on}}}{D^2} q(t)\right) \tag{1}$$

where, R_{on} and R_{off} represent the minimum and maximum resistance, respectively. μ_v denotes the dopant mobility, and D is the total thickness of the doped and undoped regions.

We add this memristor to a simple chaotic circuit defined as a three-dimensional differential equations in Fig. 2, which consists of a negative resistor $-r$, a capacitor C, two inductors L_1 and L_2, and a dual-directional diodes v_d. The nonlinear element formed by the dual-directional diodes exhibits a piecewise-linear v–i characteristic, as defined in Eq. (2).

$$v_d(i_2) = \frac{r_d}{2} \left(\left| i_2 + \frac{V}{r_d} \right| - \left| i_2 - \frac{V}{r_d} \right| \right) \tag{2}$$

The circuit equations in Fig. 2 are described by the following four-dimensional nonlinear equations:

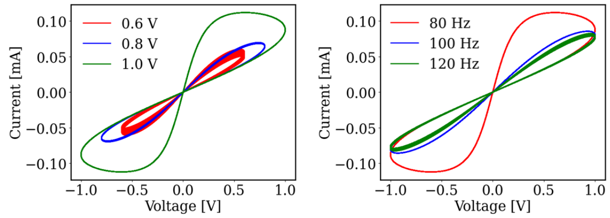
$$\begin{cases} L_1 \dfrac{di_1}{dt} = v + ri_1 - M(q)i_1 \\ L_2 \dfrac{di_2}{dt} = v - v_d(i_2) \\ C\dfrac{dv}{dt} = -i_1 - i_2 \\ \dfrac{dq}{dt} = i_1. \end{cases} \tag{3}$$

By the following normalizations,

$$i_1 = \sqrt{\frac{C}{L_1}} Vx\,,\; i_2 = \frac{\sqrt{L_1 C}}{L_2} Vy\,,\; v = Vz\,,\; q = CVw\,,$$

$$t = \sqrt{L_1 C}\tau\,,\; '\cdot' = \frac{d}{d\tau}\,,\; r\sqrt{\frac{C}{L_1}} = \alpha\,,\; \frac{L_1}{L_2} = \beta\,,$$

$$r_d \frac{\sqrt{L_1 C}}{L_2} = \gamma\,,\; R_{\mathrm{off}} \sqrt{\frac{C}{L_1}} = \eta\,,\; \frac{R_{\mathrm{on}}}{R_{\mathrm{off}}} = \zeta\,,\; \mu_v \frac{R_{\mathrm{on}}}{D^2} CV = \xi,$$



(a) (b)

Fig. 1. v-i characteristics of the memristor when a sinusoidal voltage is applied. (a) Frequency is fixed at 100 Hz and amplitude is 0.6 V, 0.8 V, and 1.0 V; (b) Amplitude is fixed at 1.0 V and frequency is 80 Hz, 100 Hz, and 120 Hz.

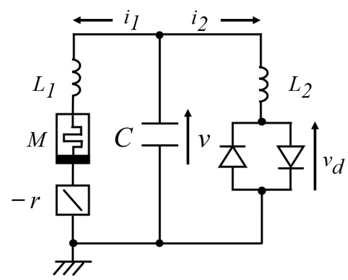


Fig. 2. Chaotic circuit with the memristor.

the normalized circuit equations are given as follows:

$$\begin{cases} \dot{x} = z + \alpha x - \eta x(\zeta \xi w + 1 - \xi w) \\ \dot{y} = z - \dfrac{\gamma}{2}\left(\left|y + \dfrac{1}{\gamma}\right| - \left|y - \dfrac{1}{\gamma}\right|\right) \\ \dot{z} = -x - \beta y \\ \dot{w} = x. \end{cases} \tag{4}$$

where α is the negative resistance, β is the ratio of inductance, γ is the resistance of the nonlinear resistor when the diodes are off, η is the maximum memristance, ζ is the minimum memristance, and ξ corresponds to the average drift mobility of the charges.

In the computer calculation, the step size of the Runge-Kutta method is set to h = 0.002. Some of the parameters are fixed to β = 2.92, γ = 456, and ξ = 0.00276.

III. Results

A. Dynamic analysis

The dynamic behavior of the circuit shown in Fig. 2 is investigated through numerical simulation over the time interval $\tau = 0$ to $10{,}000$, with the initial condition set to $(-0.016,\ -0.012,\ -0.008,\ 0)$. As a result, the circuit exhibits temporal switching between periodic and chaotic oscillations. This phenomenon differs from conventional intermittent chaos, which occurs at specific values of a bifurcation parameter. In the the circuit shown in Fig. 2, the switching is observed over a relatively wide range of bifurcation parameter values α. Figures. 3 and 4 show plots on the x–z plane and time series waveforms corresponding to two different cases: switching

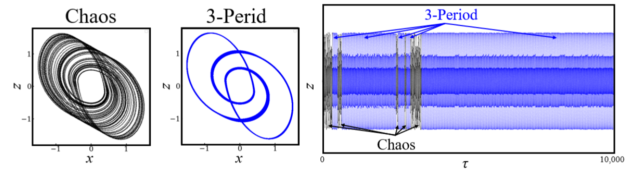


Fig. 3. Switching between 3-periodic and chaotic oscillations (α = 0.524, η = 0.0963, and ζ = 0.111).

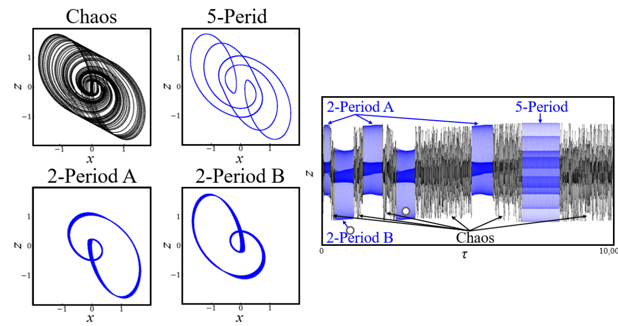
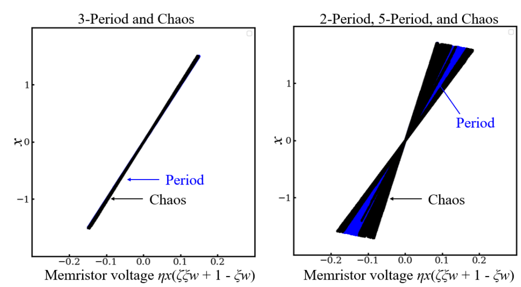


Fig. 4. Switching between 2-periodic, 5-periodic, and chaotic oscillations (α = 0.588, η = 0.0856, and ζ = 0.125).

between 3-periodic and chaotic oscillations, and switching between 2-periodic, 5-periodic, and chaotic oscillations. Figure 5 also presents the v–i characteristics of the memristor during the circuit runs. According to Eq. (4), the voltage applied to the memristor is given by $\eta x(\zeta \xi w + 1 - \xi w)$, and the current is simply x.



(a) (b)

Fig. 5. v-i characteristics of the memristor during the circuit runs. (a) 3-periodic and chaotic oscillations. (b) 2-periodic, 5-periodic, and chaotic oscillations.

As can be seen from Figs. 3 and 4, the switching between periodic and chaotic oscillations occurs in a seemingly chaotic manner. In the case of Fig. 3, 90.6% of the total duration corresponds to periodic oscillation, and Fig. 5(a) shows that the range of memristance variation is narrow. This implies that the internal state variable w in Eq. (4) changes in a relatively stable and limited range. On the other hand, in the case shown in Fig. 4, periodic oscillations account for 40.5% of the total duration. Fig. 5(b) indicates a wider variation in memristance, suggesting that w exhibits more dynamic behavior.

B. Relationship between bifurcation and the memristor

In the circuit shown in Fig. 2, the parameter α serves as the bifurcation parameter. As discussed in the previous

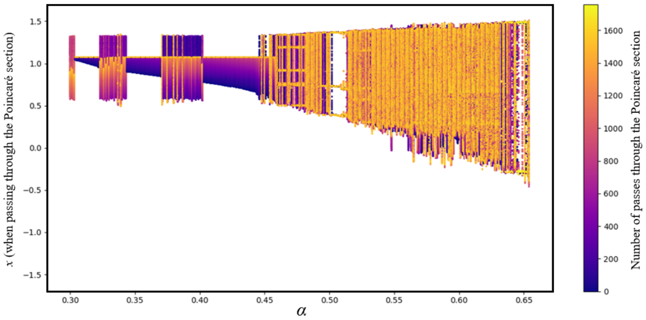


Fig. 6. Time-evolving one-parameter bifurcation diagram for α.

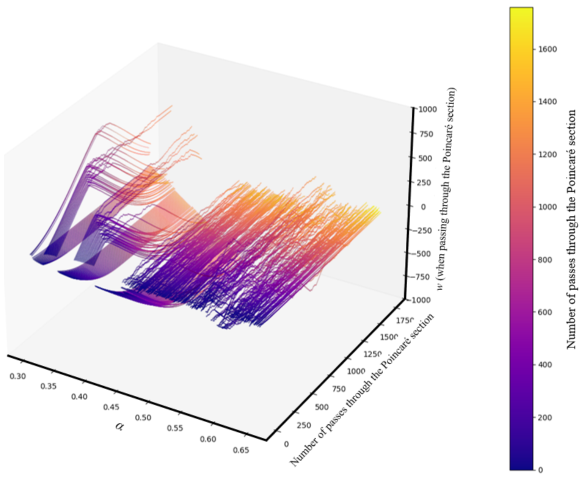


Fig. 7. Time-evolving of w corresponding to each α.

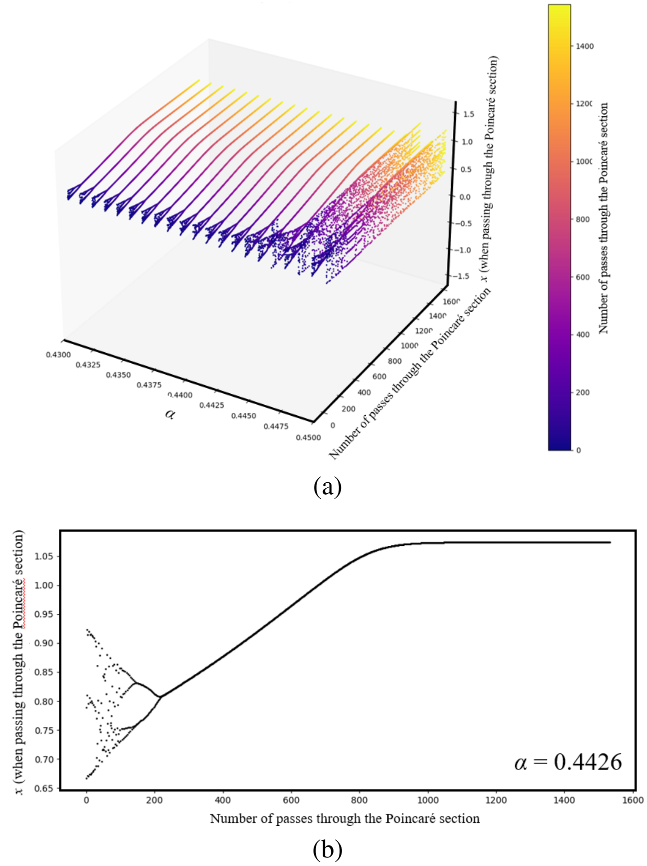


Fig. 8. Trajectory of x. (a) $\alpha = 0.4300$ to 0.4500; (b) the specific case of $\alpha = 0.4426$.

subsection, the circuit exhibits temporal switching between periodic and chaotic oscillations. To analyze this behavior, we construct a one-parameter bifurcation diagram with respect to α, considering the time-evolving changes in oscillatory states.

Since this switching phenomenon is believed to arise from the effect of the memristor, we focus on the time evolution of the state variable w, which directly influences the memristance, for each value of α. A Poincaré map is employed for this purpose. The Poincaré section is defined as the region $H \subset x > 0,\ z = 0$, and the values of x and w corresponding to each α are recorded.

Figure 6 presents the one-parameter bifurcation diagram for α, reflecting the time-dependent changes in the system, while Fig. 7 illustrates the time evolution of w for each α.

As shown in Fig. 6, bifurcations occur as α is varied, and periodic windows are observed. However, unlike typical bifurcation behavior, even for values of α that initially exhibit chaotic behavior, the system may transition to periodic behavior over time—giving the appearance of newly emerging periodic windows. This can be explained by the time evolution of w shown in Fig. 7, where w does not converge to a fixed value for many α, but instead continues to vary. This suggests that the circuit's behavior is not stationary and evolves dynamically over time, even at fixed parameter values. Furthermore, to investigate the time-evolving bifurcation phenomenon in greater detail, the region from $\alpha = 0.4300$ to 0.4500 in Fig.6 was extracted. Figure 8(a) shows the trajectory of x from $\alpha = 0.4300$ to 0.4500 and Fig. 8(b) shows the specific case of $\alpha = 0.4426$.

From Fig. 8, it can be observed that the initial bifurcation state varies depending on the value of α. In the case of $\alpha = 0.4426$, the trajectory of x initially exhibits chaotic behavior, then undergoes an inverse period-doubling bifurcation, and gradually transitions from chaos to periodic motion over time.

These results suggest that the state variable w, associated with the memristor, evolves continuously over time, potentially giving rise to the time-evolving bifurcation phenomena observed in the circuit.

C. Role of the memristor in the chaotic circuit

Based on the results obtained thus far, it is inferred that the behavior of the state variable w, which is associated with the memristor, may be related to the time-evolving bifurcation phenomena observed in the circuit. Therefore, we focus on the role of the variable w in the circuit equations. As shown in Eq. (4), w appears only in the differential equation for x, specifically on the right-hand side. The bifurcation parameter of the circuit shown in Fig. 2 is denoted by α. The variable x is involved in product terms with both the memristor-related

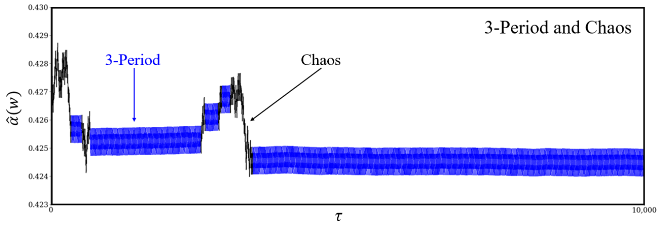


Fig. 9. Time change of $\hat{\alpha}(w)$ when the switching between 3-periodic and chaotic oscillations occurs.

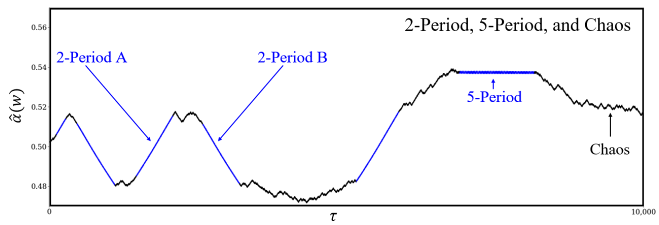


Fig. 10. Time change of $\hat{\alpha}(w)$ when the switching between 2-periodic, 5-periodic, and chaotic oscillations occurs.

variables and the parameter α. Accordingly, the differential equation for x in Eq. (4) an be reformulated with respect to x as shown in Eq. (5).

$$\begin{cases} \dot{x} = z + \alpha x - \eta x(\zeta \xi w + 1 - \xi w) \\ \quad = z + \{\eta(1-\zeta)\xi w + \alpha - \eta\}x. \end{cases} \tag{5}$$

As shown in Eq. (5), the terms α and w appear additively in a linear fashion. In the original chaotic circuit, α is treated as a fixed parameter. However, if we interpret the additive combination of α and w as a single effective parameter, the equation can be rewritten as Eq. (6).

$$\dot{x} = z + \hat{\alpha}(w)x. \tag{6}$$

We hypothesize that the time variation of $\hat{\alpha}(w)$ contributes to the switching between periodic and chaotic oscillations. Therefore, we examine the time evolution of $\hat{\alpha}(w)$.

Figures 9 and 10 show the time evolution of $\hat{\alpha}(w)$ in cases where switching occurs between 3-periodic and chaotic oscillations, and among 2-periodic, 5-periodic, and chaotic oscillations, respectively. The labels in Figs. 9 and 10 correspond to those shown in Figs. 3 and 4.

In Fig. 9, $\hat{\alpha}(w)$ varies within the range of approximately 0.424 to 0.429. When chaotic oscillations are present, $\hat{\alpha}(w)$ also exhibits chaotic fluctuations, while during 3-periodic oscillations, $\hat{\alpha}(w)$ varies periodically within a limited range.

In Fig. 10, $\hat{\alpha}(w)$ varies more widely, ranging from approximately 0.470 to 0.540. Similar to Fig. 9, $\hat{\alpha}(w)$ shows chaotic variation during chaotic oscillations. When the system exhibits 2-period A behavior, $\hat{\alpha}(w)$ increases; conversely, during 2-period B behavior, it decreases. During 5-periodic oscillations, $\hat{\alpha}(w)$ varies periodically within a bounded range.

From these results, it can be inferred that the effective bifurcation parameter value varies continuously as the state variable w evolves. This indicates that the switching between periodic and chaotic oscillations observed in the chaotic circuit with the memristor differs fundamentally from conventional intermittent chaos, which typically occurs under fixed parameter settings. Moreover, these findings suggest that the memristor may play an essential role in inducing time-evolving bifurcations in such systems.

IV. Conclusions

In this study, we investigated the dynamical behavior of a four-dimensional chaotic circuit that incorporates an HP memristor. The circuit exhibits complex oscillatory behavior including temporal switching between periodic and chaotic oscillations. Through numerical simulations, we observed that this switching phenomenon occurs not at isolated parameter values, as is typical in conventional intermittent chaos, but rather over a broad range of the bifurcation parameter α.

A key finding is that the state variable w, which governs the internal state of the memristor, evolves over time and plays a significant role in modulating the circuit's dynamics. By reformulating the circuit equations, we introduced a time-evolving effective bifurcation parameter $\hat{\alpha}(w)$, which combines the external parameter α and the internal state w. Our analysis revealed that $\hat{\alpha}(w)$ varies in a manner closely correlated with the observed switching behavior.

These results suggest that memristors enable the bifurcation parameters in chaotic circuits to vary continuously depending on the dynamics of the memristor's internal state. This behavior is fundamentally different from traditional bifurcation scenarios, where the parameter values are assumed to be constant in time. Thus, memristors not only function as nonlinear elements, but also introduce a new mechanism for generating complex time-dependent dynamics.

The insights gained from this study highlight the potential of memristor-based circuits in designing adaptive and dynamically rich systems, with possible applications in neuromorphic computing, signal processing, modeling of dynamic behaviors in biological systems, and nonlinear control.

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