Synchronization in Dynamical Oscillatory Networks with Non-Uniform Coupling Distributions

Yoko Uwate and Yoshifumi Nishio
Dept. of Electrical and Electronic Engineering, Tokushima University
2-1 Minami-Josanjima, Tokushima 770-8506, Japan
Email: {uwate, nishio}@ee.tokushima-u.ac.jp

***Abstract*— In this study, synchronization observed in dynamical oscillatory networks with non-uniform coupling distributions is investigated. The coupling states (on/off) of all connections are stochastically determined at every certain time with the coupling probability. We focus on the heavy tail type of coupling distribution for the network. By using computer simulations, we confirm that the dynamical network with the heavy tail type of coupling distribution can hardly achieve global synchronization.**

I. Introduction

The synchronization phenomena observed from coupled oscillators are suitable models to analyze the natural phenomena [1],[2]. Therefore, many researchers have proposed different coupled oscillatory networks and have discovered many interesting synchronization phenomena [3],[4].

Recently, synchronization in dynamical networks with time-varying topology has been extensively investigated [5],[6] instead of static networks (i.e., network connections are fixed constants). This is because the real-world complex networks change their topologies with time. In these studies, novel synchronization phenomena have been observed and theoretical approaches (such as Lyapunov function [5] and basin stability [6]) are used to explain the obtained synchronization phenomena. However, a node in a complex network is expressed by a mathematical model in most studies of synchronization of dynamical networks. Although, it is very important to use mathematical model for the dynamical networks in order to understand the synchronization states by approaching theoretical methods, we also need to consider physical models for future engineering applications.

Therefore, we have investigated synchronization in coupled electrical circuits systems with stochastic coupling [7]. In Ref. [7], we have confirmed that the novel synchronization states can be observed in polygonal oscillatory networks with on-off coupling by changing the coupling probability.

In this study, we focus on the brain networks as one of dynamical complex networks. Because, we would like to propose modeling of synchronization in brain by using coupled electrical oscillatory circuits, in order to make clear the mechanism of functional operation in brain. Structural and functional brain networks are explored using graph theory and the brain network structures have been made clear [8]. Furthermore, Song et al. have reported that the synaptic connectivity in local cortical circuits has heavy tail distribution [9]. We also apply this heavy tail characteristics of coupling distribution for the proposed system in this study. First, a schematic diagram of a brain network in Ref. [8] is used as a simple network model to understand detailed synchronization phenomena. Then, we extend the proposed network to a real brain network of the macaque visual cortex. By using computer simulations, we confirm that the dynamical network with heavy tail type of coupling distribution can hardly achieve global synchronization.

II. Proposed System

A. Network Model

A network model composed of 13 nodes and 22 edges is shown in Fig. 1. There are two important hubs in this network, "Connector hub" and "Provincial hub". The both hubs are high-degree nodes. "Connector hub" shows a diverse connectivity by connecting two sub-networks. "Provincial hub" primarily connects nodes in the same sub-network.

The coupling state (on/off) of adjacent nodes is determined stochastically. Each edge has a coupling probability (p). The network topology is updated at every certain time (τ=100). In this study, the node is expressed by van der Pol oscillator as shown in Fig. 2(a). The oscillators are coupled by a resistor (see Fig. 2(b)). Namely, two coupled oscillators tend to synchronize with in-phase state.

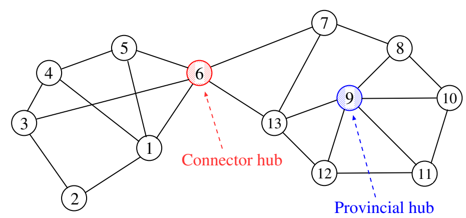
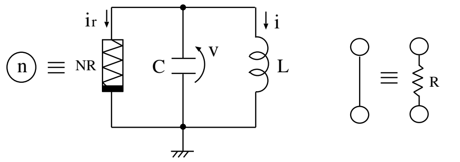


Fig. 1. Network model (node: 13, edge: 22, a schematic diagram of a brain network [8]).

Next, we develop the expression for the circuit equations of the network model. The $v_k - i_{Rk}$ characteristics of the nonlinear resistor are approximated by the following third order polynomial equation,

$$i_k = -g_1 v_k + g_3 {v_k}^3 \quad (g_1, g_3 > 0), \qquad (k = 1, 2, ...13). \tag{1}$$



(a) van der Pol oscillator. (b) Coupling method.

Fig. 2. Circuit model.

The normalized circuit equations governing the circuit are expressed as
[kth oscillator]

$$\begin{cases} \dfrac{dx_k}{d\tau} = \varepsilon\left(1 - \dfrac{1}{3}{x_k}^2\right)x_k - y_k - \gamma \displaystyle\sum_{n \in S_k} (y_k - y_n) \\ \dfrac{dy_k}{d\tau} = x_k \\ \qquad\qquad\qquad\qquad (k = 1, 2, ...13). \end{cases} \tag{2}$$

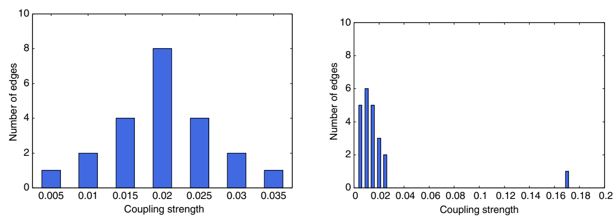
In these equations, γ is the coupling strength, ε denotes the nonlinearity of the oscillators and y_n denotes the current of oscillators connected with kth oscillator. For the computer simulations, we calculate Eq. (2) using the fourth-order Runge-Kutta method with the step size $h = 0.005$. The parameter of this circuit model are fixed as $\varepsilon = 0.1$.

B. Coupling Strength Distribution

Here, we consider three different types of coupling distribution for the network model as follows.

1) Uniform distribution
2) Gaussian distribution
3) Heavy tail distribution

In the case of Uniform distribution, all edges have the same coupling strength ($\gamma = 0.02$). Gaussian and Heavy tail distributions for the simulations are shown in Fig. 3. The total coupling strength of three different coupling distributions is set to the same value.



(a) Gaussian. (b) Heavy tail.

Fig. 3. Coupling strength distribution.

III. Synchronization Results

For the computer simulations, we simulate the network for 10^5 time steps and we fix a certain time interval (τ=100,000) for checking final synchronization state of the network. In order to analyze synchronization state, we define the synchronization as the following equation.

$$|y_k - y_n| < 0.01 \qquad (k \in S_n). \tag{3}$$

For every set of parameter values of the coupling probability, we simulate the network for 100 different distribution patterns.

A. Global Synchronization

First, we explain the simulation results of static network model (p=1.0) for applying the three coupling distributions. Table I summarizes global synchronization ratio of three coupling distributions. We observe that the networks with Uniform and Gaussian distributions achieve 100 % global synchronization. While, the network with Heavy tail distributions fails to achieve 100 % global synchronization.

TABLE I
Global Synchronization Ratio of Static Network

Coupling distribution	Global synchronization ratio [%]
Uniform	100.00
Gaussian	100.00
Heavy tail	99.10

Figure 4 shows the coupling probability dependency of global synchronization. We can see that three coupling strength distribution have different characteristics for global synchronization. When the coupling probability is small, the network with Gaussian distribution has larger global synchronization ratio than the others. The network with Uniform distribution achieve to 100 % global synchronization, when the coupling probability is set to $p = 0.4$. After that, the network with Gaussian distribution reaches to 100 % synchronization. In the case of the network with Heavy tail distribution, it can hardly become 100 % global synchronization by increasing the coupling probability.

We can say that the network with Heavy tail coupling distribution synchronizes to avoid global synchronization. The difference of global synchronization between the network with Heavy tail coupling distribution and other networks becomes large when the coupling probability is set to p=0.3 to 0.6. Namely, the coupling distribution has a huge effect on the dynamical networks which are not classified in the both of random networks (p: small) and static networks (p=1.0).

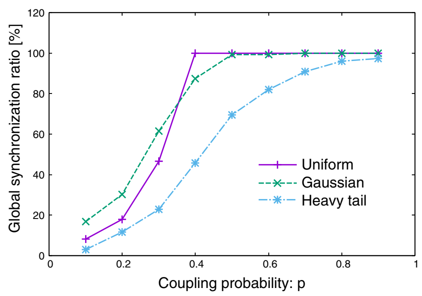
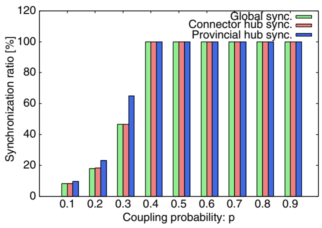


Fig. 4. Global synchronization.

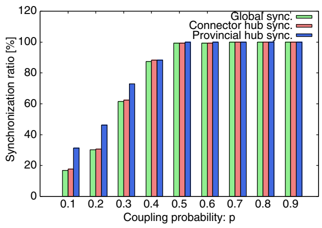
B. Cluster Synchronization

Here, we focus on cluster synchronization related with connector and provincial hubs in the network. We check the synchronization of the edges with the both hubs. If all edges of the connector hub are synchronized, we call the state connector hub synchronization. Similarly, if all edges of the provincial hub are synchronized, we call the state provincial hub synchronization.

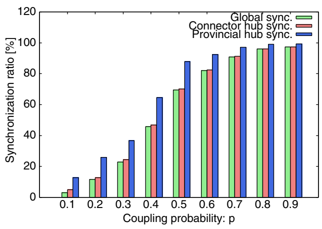
Figure 5 shows the simulation results of global, connector hub and provincial hub synchronization when three coupling strength distributions are applied to the network. In the cases of Uniform and Gaussian coupling distributions, the difference between connector and provincial hubs synchronization becomes 0 by increasing the coupling probability. While, in the case of Heavy tail coupling distribution, there are some differences between connector and provincial hubs synchronization even if the coupling probability becomes large.



(a) Uniform.



(b) Gaussian.



(c) Heavy tail.

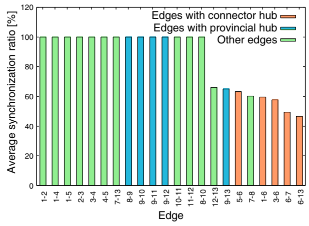
Fig. 5. Cluster synchronization.

C. Synchronization Ratio of Edge

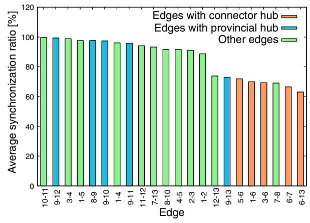
In order to investigate the effect of connector and provincial hubs, we calculate average synchronization ratio for all edges. Figure 6 shows the simulation results of three coupling strength distributions when the coupling probability is fixed with p=0.3. From this figure, we confirm that the edges connected with the provincial hub are easy to synchronize. While, the edges connected with the connector hub are difficult to synchronize. Because the synchronization ratio of edges of connector hub is smaller than the other edges.

D. Heavy Tail Distribution

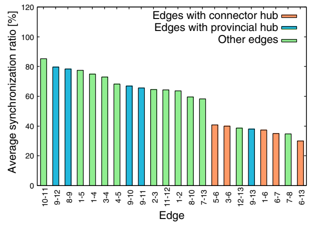
In this section, we investigate the characteristics of Heavy tail coupling distribution. We consider four patterns of Heavy tail coupling distribution as shown in Fig. 7. The position of Heavy tail is changed from γ=0.07 to 0.19 (step size: 0.04). The simulation result of global synchronization is shown in Fig. 8. From this figure, we can see that global synchronization ratio decreases by increasing the distance of the heavy tail (from pattern 1 to pattern 4). Namely, we consider that the coupling distribution of Heavy tail has important role for global synchronization in dynamical networks.



(a) Uniform.

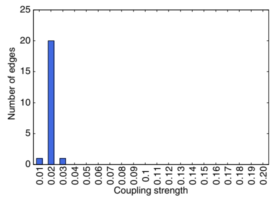


(b) Gaussian.

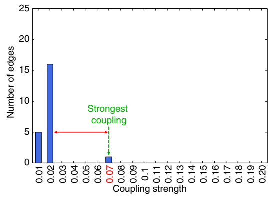


(c) Heavy tail.

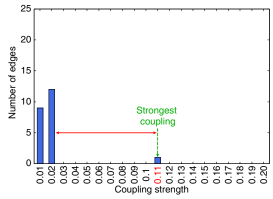
Fig. 6. Synchronization of all edges (p=0.3).



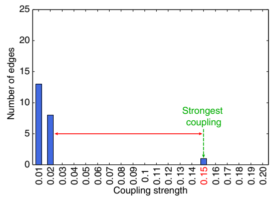
(a) Standard.



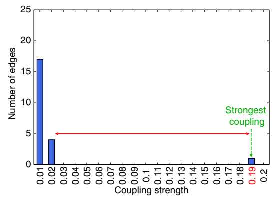
(b) Pattern 1.



(c) Pattern 2.



(d) Pattern 3.



(e) Pattern 4.

Fig. 7. Heavy tail distributions.

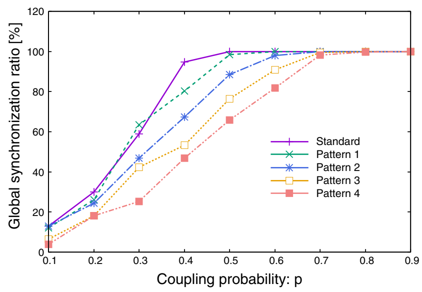


Fig. 8. Global synchronization.

IV. Applying Real Network in Brain

Finally, the proposed network is applied to real network model in brain. The modified brain network of the macaque visual cortex [10] is shown in Fig. 9. This brain network is composed of 30 nodes and 152 edges. We investigate global synchronization of the network when three different coupling distributions are applied. In the case of Uniform distribution, all edges have the same coupling strength ($\gamma = 0.002$). Gaussian and Heavy tail distributions for the simulations are shown in Fig. 10.

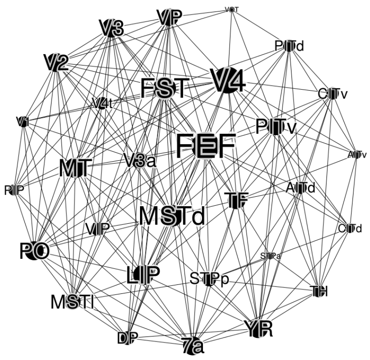
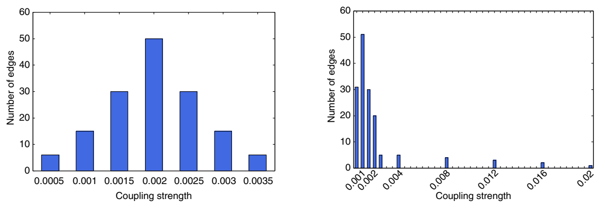


Fig. 9. Brain network of macaque visual cortex [10] (node: 30, edge: 152).



(a) Gaussian. (b) Heavy tail.

Fig. 10. Coupling strength distribution.

Table II summarizes global synchronization ratio of the static networks. We observe that the network with Uniform distribution achieves 100 % global synchronization. While, the network with Gaussian and Heavy tail distributions fail to achieve 100 % global synchronization.

TABLE II
Global Synchronization Ratio of Static Network

Coupling distribution	Global synchronization ratio [%]
Uniform	100.00
Gaussian	99.62
Heavy tail	91.11

Figure 11 shows the simulation results of global synchronization ratio. By increasing the coupling probability, global synchronization ratio of Uniform and Gaussian coupling distributions increases. While, in the case of Heavy tail coupling distribution, global synchronization ratio is smaller than other coupling distributions with whole range of the coupling probability. This results have similar characteristics with the above simple dynamical network. By applying the real brain network, we confirm that the effect of Heavy tail coupling distribution can be prominently visible for the dynamical network.

It is known that serious symptom such as epilepsia can be caused by global synchronization in brain. We consider that Heavy tail coupling distribution has important role to avoid serious symptom in normal brain.

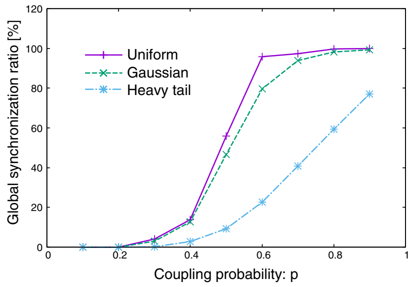


Fig. 11. Global synchronization of brain network.

V. Conclusion

We have investigated synchronization state in the dynamical oscillatory networks with non-uniform coupling distributions. We considered three different types of coupling strength; Uniform, Gaussian and Heavy tail distributions. It was confirmed that the coupling distribution of Heavy tail has different characteristics with other distributions. Namely, the dynamical network with Heavy tail coupling distribution tends to avoid global synchronization.

For the future work, we would like to consider the influence of additional noises, frequency and parameter errors in order to explain the mechanism of real brain network. Applying theoretical analysis to synchronization of the proposed network is also our future work.

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