Synchronization Phenomena in van der Pol Oscillators with Different Amplitudes

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Abstract—This article gives an explanation of the synchronization phenomena in a circuit which contains three or more van der Pol oscillators when one of them has different amplitude from the others.

1. Introduction

We are living in the world where there are so many example of synchronization: firefly luminescence, cry of birds and frogs, human applause, etc. Studies of synchronization phenomena have been reported in so many research of engineering filed: about synchronization in three coupled van der Pol oscillators with different coupling strength [1], about frustrated synchronization in coupled oscillator chains with unbalanced parametric distribution [2], synchronization-inspired partitioning and hierarchical clustering [3], about detection of synchronization phenomena in networks of hindmarsh-rose neuronal models [4], and also about synchronization in agents harvesting game with regional nature [5], etc. Furthermore, the applications of synchronization phenomena have also found in chemical, physical and biological fields: about chaos synchronization of chemical models [6], about synchronization and coupling analysis of applied cardiovascular physics in sleep medicine [7], or about multistate and multistage synchronization with excitatory chemical and electrical synapses [8], etc.

In this article, we propose a new type of coupled van der Pol oscillator system, and observe its synchronization phenomena by theoretical analysis, computer simulation and circuit experiment. In the first step, we concern about a system which contains only 3 or 4 van der Pol oscillators, then we hope to find a perfect solution of larger coupled systems.

2. Circuit Model

In this arcticle, $n > 2$ van der Pol oscillators are used to build the circuit model (Fig. 1). One van der Pol oscillator contains one capacitor (C), one inductor (L), and one nonlinear resistor.

The circuit equations are described as follows:

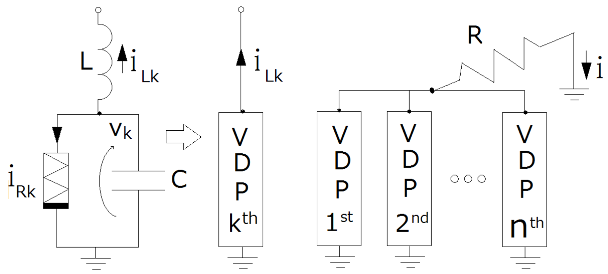


Figure 1: Circuit model.

$$C\frac{dv_k}{dt} = -i_{Lk} - i_{Rk} \quad (1)$$

$$L\frac{di_{Lk}}{dt} = -v_k - R\sum_{j=1}^{n} i_{Lk} \quad (2)$$

The $v - i$ characteristics of the nonlinear resistor are expressed by:

$$i_{Rk} = -g_1\left(\frac{v_k}{\beta_k}\right) + g_3\left(\frac{v_k}{\beta_k}\right)^3, \quad k = 1, .., n. \quad (3)$$

Figure 2 shows the relation between amplitude of the k-th oscillator and β_k. In this study, we first concern about 3 and 4 coupled oscillators system, assign the value of β_k ($k = 1, .., n-1$) as 1 by default, and change the value of β_n ($n = 3$ or 4) to observe synchronization phenomena.

3. Theoretical Analysis for n=3

To normalize circuit equations, we use Eqs. (4)-(6) to change the variables:

$$t = \sqrt{LC}\tau \quad (4)$$

$$v_k = \sqrt{\frac{g_1}{3g_3}}x_k, \; i_{Lk} = \sqrt{\frac{Cg_1}{3Lg_3}}y_k \quad (5)$$

$$\alpha = R\sqrt{\frac{C}{L}}, \; \varepsilon = g_1\sqrt{\frac{L}{C}} \quad (6)$$

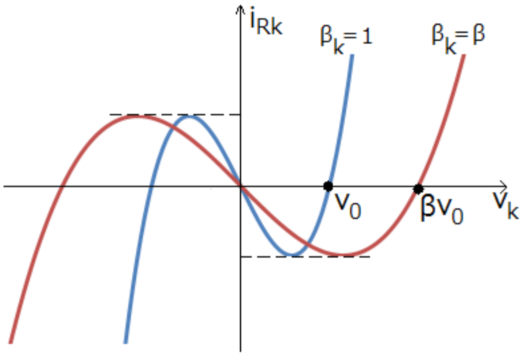


Figure 2: Relation between amplitude and β_k.

With the new variables, Eqs. (1)-(3) are normalized as:

$$\frac{dx_k}{d\tau} = \frac{\varepsilon}{\beta_k} x_k \left(1 - \frac{1}{3\beta_k^2} {x^2}_k\right) - y_k \quad (7)$$

$$\frac{dy_k}{d\tau} = x_k - \alpha \sum_{j=1}^{3} y_j \quad (k = 1, .., 3) \quad (8)$$

where α is the coupling factor and ε is the strength of non-linearity. We will simulate the above equations later, and now we first use the averaging method to analyze x_k, y_k. We assume that x_k, y_k can be considered as bellow:

$$x_k(\tau) = \rho_k(\tau)\cos(\tau + \theta_k(\tau)) \quad (9)$$

$$y_k(\tau) = \rho_k(\tau)\sin(\tau + \theta_k(\tau)) \quad (10)$$

Assign Eqs. (9)-(10) to Eqs. (7)-(8), and we obtain:

$$\dot{\rho_k} = M_k \cos\phi_k - \alpha \sin\phi_k \sum_{i=1}^{3} \rho_i \sin\phi_i \quad (11)$$

$$\dot{\theta_k} = -\frac{M_k}{\rho_k}\sin\phi_k - \alpha \frac{\cos\phi_k}{\rho_k} \sum_{i=1}^{3} \rho_i \sin\phi_i \quad (12)$$

where

$$M_k = \frac{\varepsilon}{\beta_k} f(x_k), \quad f(x_k) = x_k - \frac{1}{3{\beta_k}^2} {x_k}^3 \quad (13)$$

$$\phi_k = \tau + \theta_k \quad (14)$$

By averaging Eqs. (11)-(12) over one period, as averaging method's theory, there is a positive ε_r which small enough that satisfied: for every ε inside $(0, \varepsilon_r)$, $\rho_k(\tau)$ and $\theta_k(\tau)$ can be considered as constant and the values of $\dot{\rho_k}$, $\dot{\theta_k}$ can be calculated as:

$$\dot{\rho_k} = \frac{1}{2\pi}\int_0^{2\pi}(M_k\cos\phi_k - \alpha\sin\phi_k \textstyle\sum_{i=1}^{3}\rho_i\sin\phi_i)d\phi_k \quad (15)$$

$$\dot{\theta_k} = \frac{1}{2\pi}\int_0^{2\pi}(\frac{-M_k\sin\phi_k}{\rho_k} - \frac{\alpha\cos\phi_k}{\rho_k} \textstyle\sum_{i=1}^{3}\rho_i\sin\phi_i)d\phi_k \quad (16)$$

We obtain the following equations:

$$\dot{\rho_k} = \frac{\varepsilon}{2\beta_k}\left(\rho_k - \frac{{\rho_k}^3}{4{\beta_k}^2}\right) - \frac{\alpha}{2}\sum_{i=1}^{3}\rho_i\cos\varphi_{ki} \quad (17)$$

$$\dot{\theta_k} = \frac{\alpha}{2\rho_k}\sum_{i=1}^{3}\rho_i\sin\varphi_{ki} \quad (18)$$

where $\varphi_{ij} = \theta_i - \theta_j$ is the phase difference between i-th and j-th oscillator for $i, j = 1, 2, 3$. Because the role of the first and the second oscillator are equal in this 3 coupled oscillators system, we can assume that $\rho_1 = \rho_2 = \rho$. In steady state, $\dot{\rho_k}$ and $\dot{\theta_k}$ equal zero, then Eqs. (17)-(18) can be rewritten as:

$$\frac{1}{2}\varepsilon\rho\left(1 - \frac{\rho^2}{4}\right) - \frac{\alpha}{2}(\rho + \rho\cos\varphi_{12} + \rho_3\cos\varphi_{31}) = 0 \quad (19)$$

$$\frac{1}{2}\frac{\varepsilon}{\beta_3}\rho_3\left(1 - \frac{\rho^2}{4{\beta_3}^2}\right) - \frac{\alpha}{2}(\rho_3\cos\varphi_{31} + \rho\cos\varphi_{32} + \rho_3) = 0 \quad (20)$$

$$\rho\sin\varphi_{12} - \rho_3\sin\varphi_{31} = 0 \quad (21)$$

$$\sin\varphi_{31} + \sin\varphi_{32} = 0 \quad (22)$$

Actually:

$$\varphi_{32} = \varphi_{12} + \varphi_{31} \quad (23)$$

then Eqs. (19)-(22) have two solution.

Solution 1:

$$\rho = 0 \quad (24)$$

$$\rho_3 = 0 \quad (25)$$

$$\varphi_{31} = m\pi \quad (26)$$

$$\varphi_{12} = l\pi \quad (27)$$

Solution 2:

$$\rho = 2 \quad (28)$$

$$\rho_3 = 2\beta_3 \quad (29)$$

$$\varphi_{31} = \pm\cos^{-1}\left(-\frac{\beta_3}{2}\right) \quad (30)$$

$$\varphi_{12} = 2k\pi \mp 2\cos^{-1}\left(-\frac{\beta_3}{2}\right) \quad (31)$$

where k, m, l are integers. We use Jacobi matrix to concern about stability of each solution. For the solution 1, the Jacobi matrix contains positive eigenvalues, and it is unstable solution, but for the solution 2, the Jacobi matrix contains only negative eigenvalues, consequently it is stable solution. In other words, solution 2 is the theoretical analysis results of this 3 coupled oscillators system. The graph of this theoretical analysis results is shown in Fig. 3.

In this article, we assume that β_3 is changed inside [0.2, 2] only but actually, when β_3 is lager than 2, Eq. (30) is not mathematically correct, and in this case, phase difference φ_{31} and φ_{12} are both zero.

4. Synchronization for $n = 3$

For the case of $n = 3$, $\beta_1 = \beta_2 = 1$, Fig. 4 shows three of computer simulation results obtained by using Runge-Kutta method when β_3 equals 0.1, 1.5 and 2.0. In this

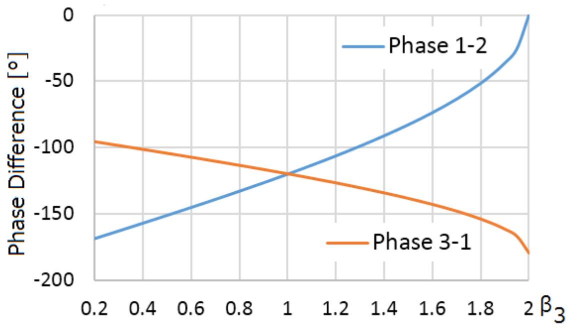


Figure 3: Theoretical analysis results when n = 3.

figure, there are $x_i - x_j$ windows show the phase relation between a pair of oscillators, while the x_1, x_2, x_3 windows show the time waveform of each single oscillator.

The phase differences are summarized in Fig. 5. By comparing Fig. 3 and Fig. 5, the theoretical analysis results and computer simulation results agree well. Furthermore, there is not α in the theoretical analysis results. To explain this, we can calculate the current of resistor R by calculating $\sum_{k=1}^{3} y_k(t)$. By using Eqs. (28)-(29), we obtain:

$$y_i = 2\beta_i \sin(\tau + \theta_i), \; \beta_1 = \beta_2 = 1 \tag{32}$$

then:

$$\sum_{k=1}^{3} y_k(\tau - \theta_3) = 2\sin(\tau - \varphi_{31}) + 2\sin(\tau + \varphi_{23}) + 2\beta_3 \sin(\tau) \tag{33}$$

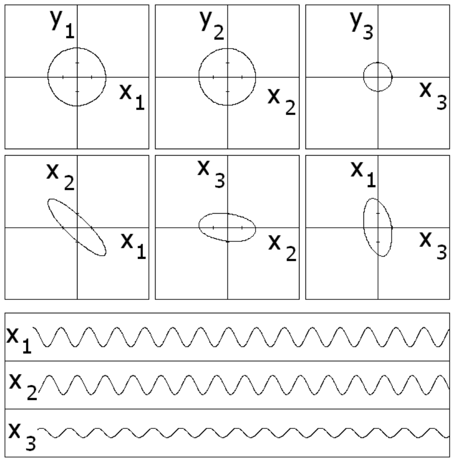
Substituting Eqs. (28)-(31) into Eq. (32), we obtain:

$$\sum_{k=1}^{3} y_k(\tau) = 0 \tag{34}$$

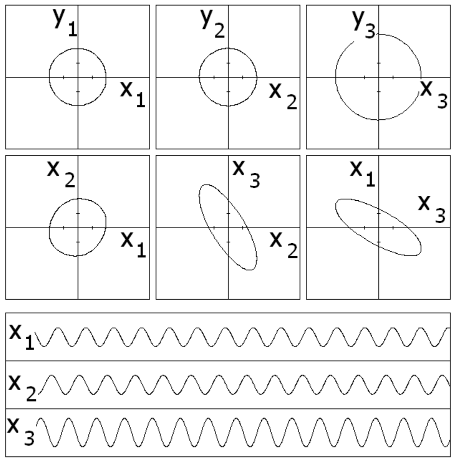
this corresponds $i(t) = 0$ at every value of t. In other words, the electric current of resistor R is always zero whenever synchronization phenomena occurs. Figure 6 shows one of our circuit experimental results. In Fig. 6, we obtain (φ_{12}, φ_{23}, φ_{31}) equals (−150°, −105°, −105°). By comparing these values to Figs. 4-5 above, it is shown that Fig. 6 expresses for the circuit experimental results when ($\beta_1, \beta_2, \beta_3$) are about (1, 1, 0.5).

5. Synchronization for n = 4

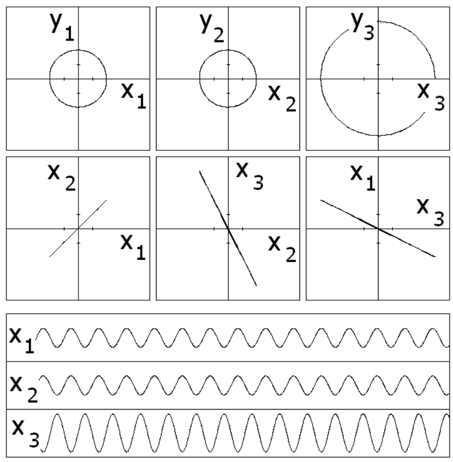
For the case of n = 4, $\beta_1 = \beta_2 = \beta_3 = 1$, Fig. 7 shows the computer simulation results when β_4 equals 3.0. In Fig. 7, the first three oscillators which have the same amplitude reach the in-phase, while each one of them and the fourth oscillator reach the anti-phase synchronized oscillatory state.



(a) β_3 = 0.5.



(b) β_3 = 1.5.



(c) β_3 = 2.0.

Figure 4: Computer simulation results when n = 3.

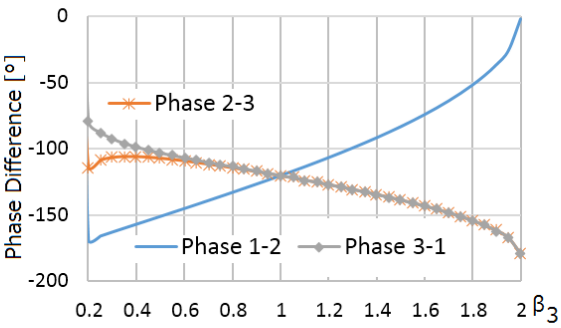


Figure 5: Computer simulation results when $n = 3$.

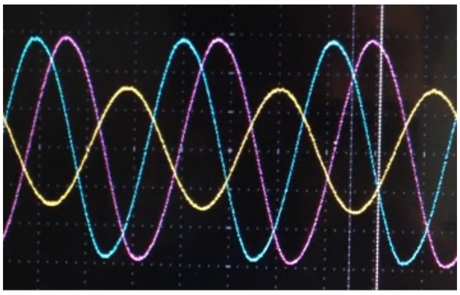


Figure 6: Circuit experimental results when $n = 3$.

6. Conclusions

This study mainly concerned about the synchronization phenomena, especially about phase difference between the oscillators when the amplitudes are not equal. However, there is slight difference between computer simulation results and theoretical analysis results when β_3 is small, and the reason of this difference has not been perfectly explained. In the next step, it is necessary to increase the number of ocsillators and complete the theoretical analysis when n is larger than 4, and it is expected to bring us more interesting phenomena.

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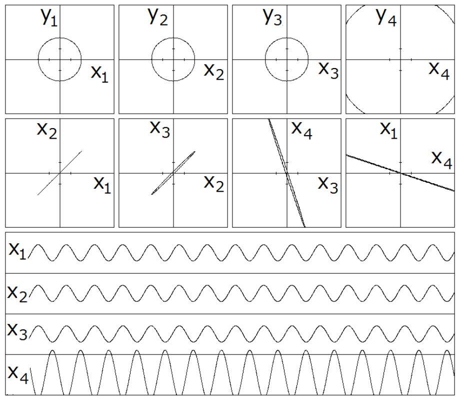

Figure 7: Computer simulation results when $n = 4$, $\beta_4 = 3$.

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