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# Synchronization Phenomena in Coupled van der Pol Oscillators with Nonlinearity Errors

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## Abstract

In this study, we investigate synchronization phenomena observed in completely coupled ten van der Pol oscillators. Ten oscillators are divided into three groups. Each group consists of two, three and five oscillators. When the nonlinearity of each oscillator has different values, we observe the interesting synchronization phenomena in the circuit system.

## 1. Introduction

Synchronization phenomena can be observed everywhere in natures such as firefly luminescences, cry of frogs, cardiac heartbeat and so on. Coupled oscillators are often used to investigate the synchronization phenomenon. Also studies on synchronization phenomena of coupled oscillators are extensively carried out in various fields, physics, biologies, engineerings and so on [1], [2]. Therefore, many types of the coupled circuit models using van der Pol oscillators are proposed to understand synchronization phenomena. Endo et al. have reported the details of a theoretical analysis and the corresponding circuit experiments on electrical oscillators arranged in a ladder, a ring and a two-dimensional array topologies [3]-[5]. Moreover, coupled oscillatory systems can also produce interesting phase patterns, including wave propagation, clustering and complex patterns. Setou et al. have observed interesting synchronization phenomena (oscllation death, independent oscillation and double mode oscillation) when van der Pol oscillators with different frequencies are coupled by means of a resistor in a star topology [6]. We assume that any frustrations such as different frequencies have one possibility to produce interesting synchronization phenomena in oscillatory systems. However, there are not many discussions the coupled oscillatory systems with nonlinearity errors.

In this study, we investigate synchronization phenomena observed in completely coupled ten van der Pol oscillators, when ten van der Pol oscillators are divided into three groups. Each oscillator has different values of the nonlinearity depending on the error parameters. By using computer simulations, we focus on the phase difference between the coupled oscillators in each group. We observe interesting synchronization phenomena when the nonlinearity error is changed.

## 2. Circuit model

Figure 1 shows the conceptual circuit model of our investigation. Ten van der Pol oscillators are completely coupled and divided into three groups. Group1 is composed of 1st and 2nd oscillators. Group2 is composed of 3rd, 4th and 5th oscillators. Other oscillators belong to group3.

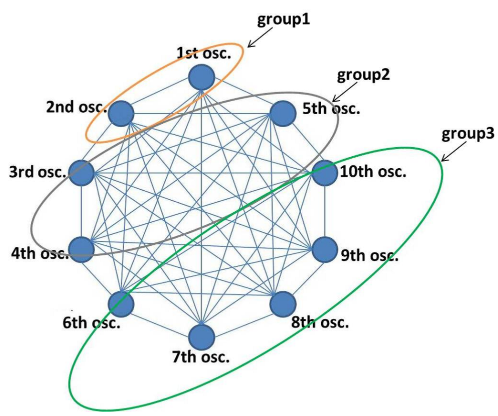
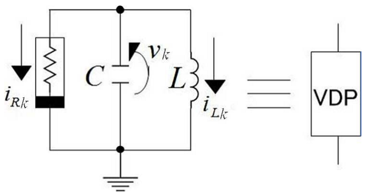



Figure 1: Conceptual circuit model.

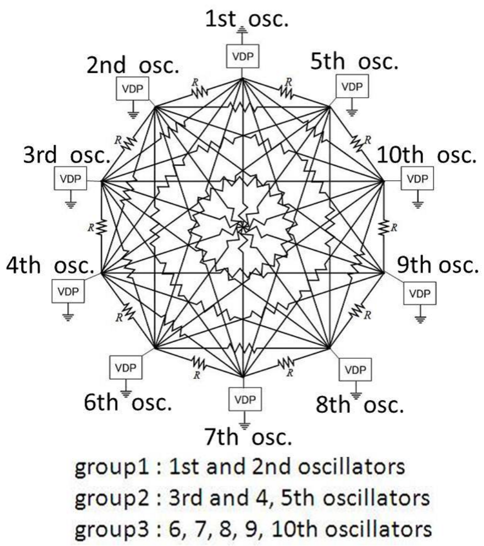
The electrical circuit model of van der Pol oscillator is shown in Fig. 2(a). Figure 2 shows the circuit model when ten van der Pol oscillators are completely coupled by resistors.

The $v_k - i_{Rk}$ characteristics of the nonlinear resistor are approximated by the following equation.

$$i_{Rk} = -g_1 v_k + g_3 v_k^3 \quad (k = 1, 2, ..., 10), \tag{1}$$




(a) van der Pol oscillator.




(b) Completely coupled ten van der Pol oscillators.

Figure 2: Circuit model.

By using the variables and the parameters,

$$t = \sqrt{LC}\tau,\ \ v_k = \sqrt{\frac{g_1}{g_3}}x_k,\ \ i_k = \sqrt{\frac{g_1}{g_3}}\sqrt{\frac{C}{L}}y_k$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}},\ \gamma = \frac{1}{R}\sqrt{\frac{L}{C}},$$

The normalized circuit equations of ten coupled van der Pol oscillators are given as follows:

$$\begin{cases} \dfrac{dx_k}{d\tau} = \varepsilon(1 - x_k^2)x_k - y_k - \gamma \displaystyle\sum_{k=1}^{10} x_k \\ \dfrac{dy_k}{d\tau} = x_k. \end{cases} \tag{2}$$

where $\varepsilon$ is the nonlinearity and $\gamma$ is the coupling strength.

## 3. Simulation results

For the computer simulations, we calculate Eq. (2) using a fourth-order Runge-Kutta method with step size $h$ = 0.005. The parameter of the standard nonlinearity is fixed with $\varepsilon$ = 1.00, and the coupling strength is set to $\gamma$ = 0.01. In this section, we investigate synchronization state when the error is added to the nonlinearity of each oscillator. We define the error rate of each oscillator as follows:

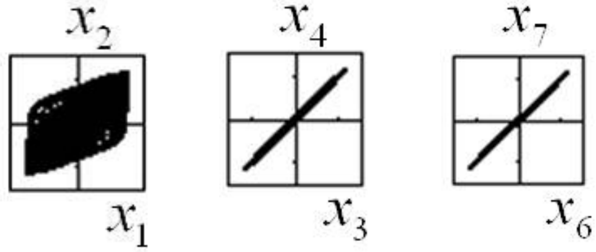
$$\begin{cases} \varepsilon_1 = \varepsilon \\ \varepsilon_2 = \varepsilon + \delta \\ \varepsilon_3 = \varepsilon + \delta + g \\ \varepsilon_4 = \varepsilon + 2\delta + g \\ \varepsilon_5 = \varepsilon + 3\delta + g \\ \varepsilon_6 = \varepsilon + 4\delta + 2g \\ \varepsilon_7 = \varepsilon + 5\delta + 2g \\ \varepsilon_8 = \varepsilon + 6\delta + 2g \\ \varepsilon_9 = \varepsilon + 7\delta + 2g \\ \varepsilon_{10} = \varepsilon + 8\delta + 2g, \end{cases} \tag{3}$$

where, $\delta$ denotes the small error and $g$ denotes the large error of the nonlinearity. We set the range of the parameters $\delta$=[0.00:0.030], $g$=[0.00:0.30].

First, we investigate synchronization phenomena in ten coupled oscillators when the nonlinearity errors are fixed with $\delta$ = 0.01, $g$ = 0.00. In this case, we confirm that all oscillators are synchronized at in-phase state.

Next, we investigate the synchronization phenomena when the nonlinearity errors are fixed with $\delta$ = 0.01, $g$ = 0.22. Figure 3 shows the phase difference between two oscillators in each group. We can see that the phase difference between 1st and 2nd oscillators in group1 is changing and has amount of the phase shift (Fig. 3 (a)). While, 3rd/4th and 6th/7th oscillators in group2 and group3 are synchronized at in-phase as in Figs. 3 (a),(b).

Figure 4 shows the phase difference between 1st and 2nd oscillators with iteration time. From this figure, we observe the switching phenomenon of the phase difference. The phase difference changes from 5 to 70 degrees periodically.




(a) 1st-2nd (b) 3rd-4th (c) 6th-7th

Figure 3: Phase difference between two oscillators in each group ($\delta$ = 0.01, $g$ = 0.22).

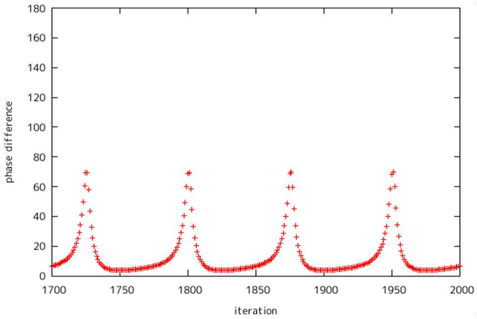
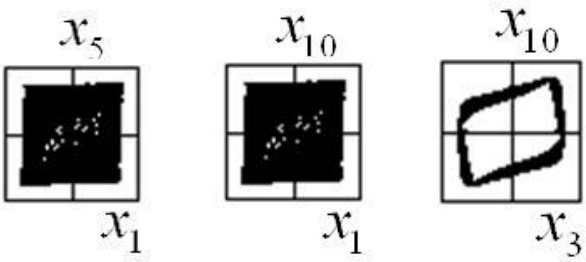



Figure 4: Phase difference of 1st and 2nd oscillators. ($\delta$ = 0.01, $g$ = 0.22).

Figure 5 shows the phase difference between the groups for $\delta$ = 0.01, $g$ = 0.22. The synchronization state of oscillators between group1 and group2 or group1 and group3 are asynchronous (Figs. 5 (a), (b)). The oscillators between group2 and group3 are synchronized with amount of phase shift as shown in Fig. 5 (c). From Fig. 5, we can see that the synchronization states between first and second oscillators in group1 are affected by group2 and group3.



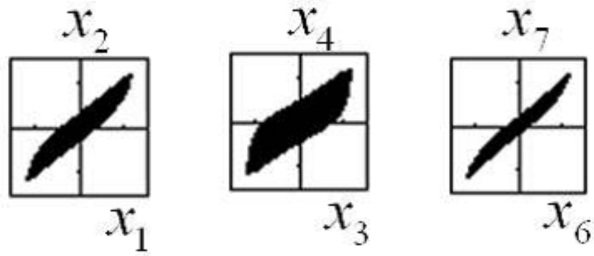

(a) group1-group2 (b) group1-group3 (c) group2-group3

Figure 5: Phase difference between groups ($\delta$ = 0.01, $g$ = 0.22).

By increasing the value of $g$, we observe the different type of synchronization phenomena. In Fig. 6, the results of phase differences between two oscillators in each group for $\delta$ = 0.01, $g$ = 0.30 are shown. The two oscillators in each group are almost synchronized at in-phase state. Figure 7 shows the phase difference between first and second oscillators with iteration time. We can see that the aperiodic oscillation of the phase difference from 5 to 20 degrees.

Figure 8 shows the phase difference between the groups for $\delta$ = 0.01, $g$ = 0.30. In this case, we confirm that the synchronization state of oscillators between groups are asynchronous for all cases.

Next, we focus on synchronization phenomena depending on the parameter dependency. Figure 9 shows the phase difference between 1st and 2nd oscillators when the parameters




(a) 1st-2nd (b) 3rd-4th (c) 6th-7th

Figure 6: Phase difference between two oscillators in group ($\delta$ = 0.01, $g$ = 0.30).

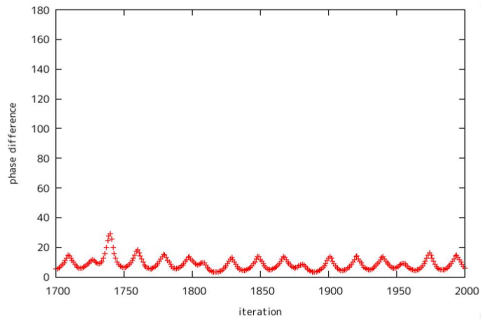
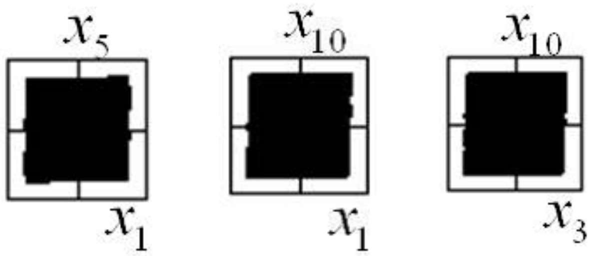



Figure 7: Phase difference of 1st and 2nd oscillators. ($\delta$ = 0.01, $g$ = 0.30).

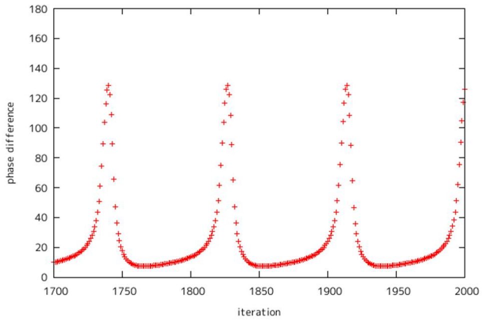



(a) group1-group2 (b) group1-group3 (c) group2-group3

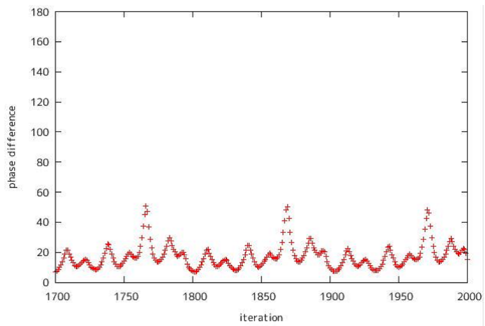
Figure 8: Phase difference between groups ($\delta$ = 0.01, $g$ = 0.30).

of the nonlinearity error $\delta$ and $g$ are changed. By increasing the value of $\delta$, the oscillation frequency of the phase difference becomes slow and the range of the phase shift becomes large (see Fig. 4 and Fig. 9 (a)). In comparison between Fig. 7 and Fig. 9 (b), we can see that the range of the phase difference becomes large.

Finally, the sojourn time of in-phase state and one-period is measured by changing the nonlinearity error $g$. In this simulation, in-phase state is defined when the value of the phase difference smaller than 15 degrees. One-period means the sim-

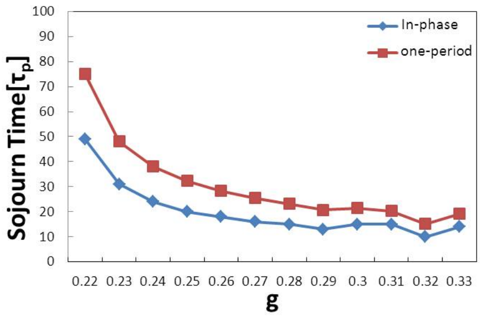


(a) $\delta = 0.02$, $g = 0.19$.

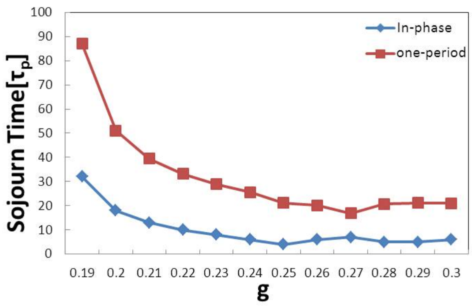


(b) $\delta = 0.02$, $g = 0.30$.

Figure 9: Phase difference of first and second oscillators.




(a) $\delta = 0.01$.




(b) $\delta = 0.02$.

Figure 10: Sojourn time of the in-phase state.

ulation time between the peaks of the phase difference. The simulation results for $\delta$=0.01 and 0.02 are shown in Fig. 10. The horizontal axis is the nonlinearity error $g$ and the vertical axis is the sojourn time ($\tau_p$). From these results, we confirm that the sojourn time of in-phase and one-period decreases with the nonlinearity error $g$.

## 4. Conclusions

In this study, we have investigated synchronization phenomena observed in completely coupled ten van der Pol oscillators, when ten van der Pol oscillators are divided into three groups. Each oscillator has different values of the nonlinearity depending on the error parameters. By using computer simulations, we have observed interesting synchronization phenomena when the nonlinearity error is changed.

For the future work, we would like to investigate the obtained synchronization phenomena in detail and apply this proposed circuit system to general large-scale networks.


### Acknowledgment

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