Synchronization of Coupled Tetrahedral Oscillatory Networks Shared by Triangular Face

Takahiro Nagai, Hironori Kumeno, Yoko Uwate and Yoshifumi Nishio
Dept. of Electrical and Electronic Engineering, Tokushima University,
2-1 Minamijosanjima, Tokushima, 770-8506 Japan
Email: {nagataka, kumeno, uwate, nishio}@ee.tokushima-u.ac.jp

***Abstract*—In this study, we investigate the synchronization phenomena observed in two coupled tetrahedral oscillators shared triangular face. We focus on all of the coupling strength between adjacent oscillators and we show the change of the phase differences when we change the coupling strength. Next, we vary the coupling strength partially and observe synchronization states of shared triangular oscillators.**

I. Manuscript Submission

Coupled oscillatory system is one of the most proper model to indicate the high-dimensional phenomena in natural science fields [1]-[7]. There are a lot of synchronization phenomena in natural environment. For example, swing of the pendulum, firefly luminescence, cardiac heartbeat etc. are well known as the synchronization phenomena. Therefore investigations of the coupled oscillatory systems about synchronization phenomena are reported in biology, physics and mathematics. Coupled oscillatory systems have various wave patterns and including wave propagations, clustering and complex patterns. Hence, various types of coupled oscillators were proposed, and the part of the mechanism of the non-linear phenomenon has been elucidated until now. However investigations on the complex coupled oscillatory networks have not been studied enough yet.

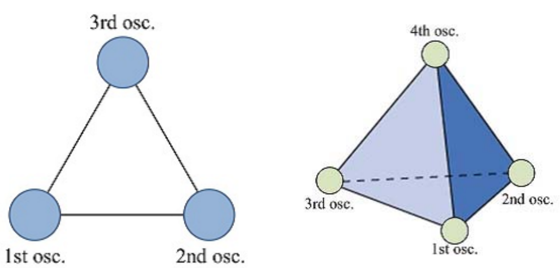

Fig. 1. Coupled van der Pol oscillators. (a) Three coupled oscillator. (b) Four coupled oscillator.

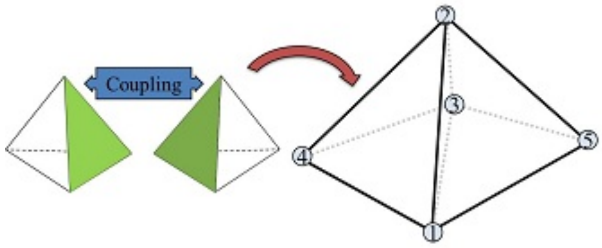

Fig. 2. Conceptual circuit model for tetrahedron form.

In our investigations, we use several van der Pol oscillators. Van der Pol oscillators have been coupled in various form and investigated about their synchronization phenomena [8], [9]. In particular, three coupled oscillatory system with a ring topology as shown Fig. 1 have made the very interesting results [10]. In this circuit system, each oscillator was coupled by an inductor and the number of coupled oscillators was an odd number. Then, we could not observe the synchronization phenomena with in/anti-phase states. That is to say, three-phase synchronization (phase shift: 120°) is obtained for the case of three oscillators by the effect of frustration. However, the three-phase synchronization was always observed stably in that system.

In our previous study, we have investigated several kinds of interesting synchronization phenomena in coupled oscillatory system which has stronger frustrations. We have researched four coupled van der Pol oscillators in the regular tetrahedron form as shown in Fig. 1(b). By computer simulation, we observed that the phase difference between adjacent oscillators changed and the synchronization was destroyed after the adjacent oscillators synchronize with anti-phase. In other words, this circuit model has the feature such as repeated anti-phase synchronous and asynchronous. The another study, synchronization phenomena in two coupled triangular oscillatory networks sharing a branch was investigated in [11]. In this case, we could observed synchronization that the phase difference of sharing branch is in-phase (phase difference: 0°), and the other one between adjacent oscillators are synchronized with anti-phase (phase difference: 180°). In this study, we investigate synchronization phenomena in coupled oscillatory system which has two tetrahedrons. We consider two tetrahedron forms sharing a triangular face as shown in Fig. 2. We carry out the computer simulation and analyze the synchronization phenomena.

First we investigate phase difference between adjacent oscillators and compare shared triangular oscillators to the other oscillators. Next, we vary the coupling strength partially and observe synchronization states of shared triangular oscillators.

II. Circuit Model

The circuit model is shown in Fig. 3(a). In this circuit model, two tetrahedrons oscillators are coupled by the triangular face and the fourth and the fifth oscillators have no connection. In the computer simulations, we assume that the $v_k - i_{Rk}$ characteristics of nonlinear resistor in each oscillator is given by the following third order polynomial equation.

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3 \qquad (g_1,\ g_3 > 0), \quad (k = 1,\ 2,\ 3,\ 4,\ 5). \tag{1}$$

The normalized circuit equations are expressed as:

$$\begin{cases} \dfrac{dx_k}{d\tau} = \varepsilon(1 - \dfrac{1}{3}{x_k}^2)x_k - (y_{ak} + y_{bk} + y_{ck} + y_{dk}) \\ \dfrac{dy_{ak}}{d\tau} = \dfrac{1}{4}\big\{x_k - \eta y_{ak} - \gamma(y_{ak} + y_n)\big\} \\ \dfrac{dy_{bk}}{d\tau} = \dfrac{1}{4}\big\{x_k - \eta y_{bk} - \gamma(y_{bk} + y_n)\big\} \\ \dfrac{dy_{ck}}{d\tau} = \dfrac{1}{4}\big\{x_k - \eta y_{ck} - \gamma(y_{ck} + y_n)\big\} \\ \dfrac{dy_{dk}}{d\tau} = \dfrac{1}{4}\big\{x_k - \eta y_{dk} - \gamma(y_{dk} + y_n)\big\}. \end{cases} \tag{2}$$

We use the following normalizations:

$$t = \sqrt{LC}\tau \ , v_k = \sqrt{\frac{g_1}{3g_3}}\, x_k \ , i_{ak} = \sqrt{\frac{g_1 C}{3g_3 L}}\, y_{ak},$$

$$i_{bk} = \sqrt{\frac{g_1 C}{3g_3 L}}\, y_{bk} \ , i_{ck} = \sqrt{\frac{g_1 C}{3g_3 L}}\, y_{ck},$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}} \ , \gamma = R\sqrt{\frac{C}{L}} \ , \eta = r_m\sqrt{\frac{C}{L}},$$

(k= 1, 2, 3, 4, 5),

where ε is the nonlinearity, γ is the coupling strength,

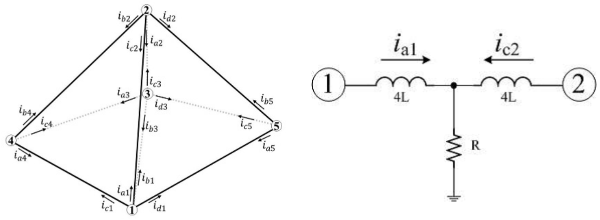

Fig. 3. Circuit model with double tetrahedrons. (a) Conceptual circuit model. (b) Coupled structure.

η indicates the resistive component and y_n denotes the current of neighbor oscillator on coupling resistor. In the computer simulations, we calculate the phase differences between adjacent oscillators. Here r_m denotes the internal resistance of an inductor.

III. Synchronization Phenomena

We calculate Eq. (2) using the fourth-order Runge-Kutta method with the step size $h = 0.01$. We show the simulation result of the synchronization phenomena in Fig. 4. In this figure, we show the attractor of each oscillator and the horizontal axis is the voltage of each oscillator, and the vertical axis is the electric current of each oscillator. These electric currents are summed the four currents $y_k = y_{ak} + y_{bk} + y_{ck} + y_{dk}$ and we set the parameters ε = 0.10, γ = 0.10 and η = 0.00010.

Next, the time waveforms of the voltage of each capacitor after sufficient time has elapsed are shown in Fig. 5. And the phase differences between the adjacent oscillator of this case is equal to the result as shown in Fig. 6. As a result, in the case of this circuit model, it was observed that the phase difference finally converged with the constant value. Also, this phase difference hardly varies even if we change the initial conditions.

But the effect of varying initial conditions are able to

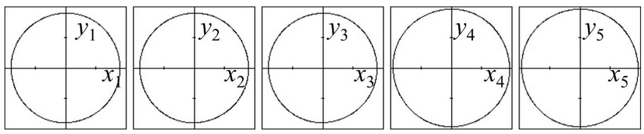

Fig. 4. Attractor between adjacent oscillators (horizontal axis:x_k, vertical axis:y_k $(k = 1,\ 2,\ 3,\ 4,\ 5)$.

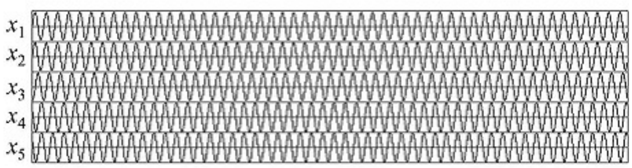

Fig. 5. The time waveform of the each oscillator.

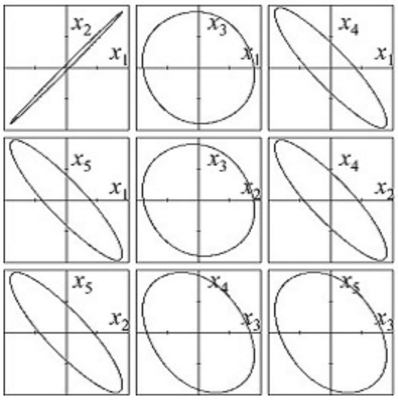

Fig. 6. Lissajous figures.

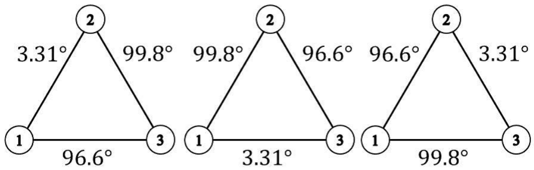

Fig. 7. The relationship of phase differences for the shared triangle.

confirm other point of view. The pattern of phase differences between shared three oscillators are rotated by the initial conditions. Figure 7 shows the phase differences between shared triangular oscillators. In this figure, we

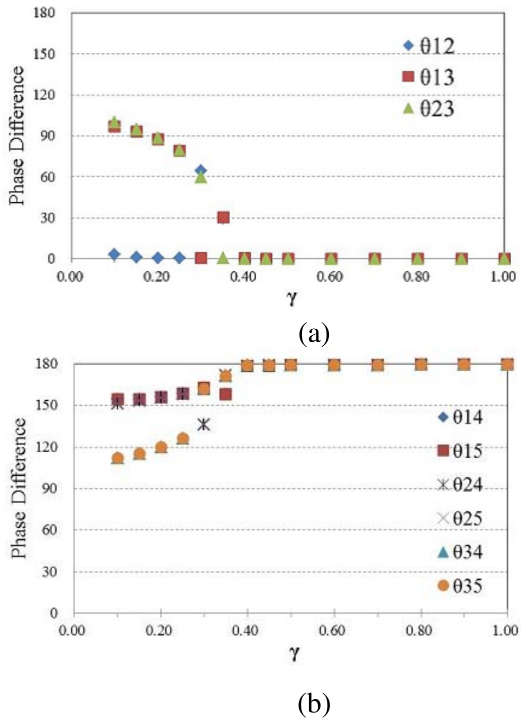

Fig. 8. The change of phase differences by coupling strength. (a) shared triangular oscillators. (b) other oscillators.

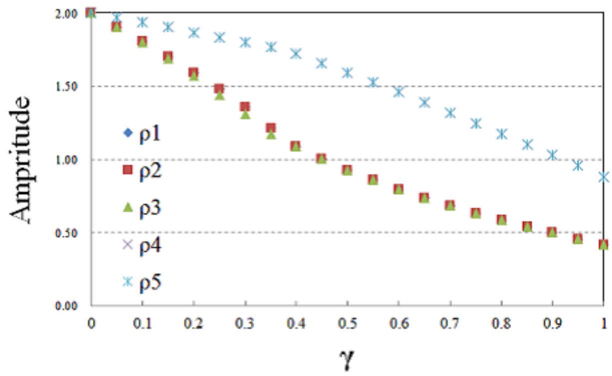

Fig. 9. Amplitude of voltage.

can find that the three patterns of synchronization states exit.

Here, we change the coupling strength from $\gamma = 0.10$ to 1.00. In this case, phase differences divide into two groups and synchronize with in/anti-phase states. Shared triangular oscillators synchronize with in-phase but other synchronization states between adjacent oscillators shifts with anti-phase. We summarize the changes of the phase difference for the coupling strength in Fig. 8. We show the phase difference of shared oscillators (Fig. 8(a)), and the other oscillators (Fig. 8(b)). When the coupling strength is strong, the phase differences converge for in/anti-phase. Also, We describe changes of the amplitude ρ_k of voltage in Fig. 9. In this result, the amplitudes are divided into two groups that one of the groups is the shared oscillators and the other one is the remained oscillators.

IV. Change of Coupling Strength

In this circuit system, if we break the coupling about forth and fifth oscillators, the three coupled oscillators with ring topology exists. Namely, we can observe the three phase synchronization between adjacent oscillators and other synchronization states are asynchronous. Here we fix the coupling strength $\gamma = 0.100$, we multiply β $(0 < \beta < 1.0)$ coupling strength about forth and fifth oscillator. We observe change of the synchronization states between adjacent oscillators. Table 1 shows the synchronization states by varying β.

TABLE I
THE CHANGE OF SYNCHRONIZATION STATES BY VARYING β.

β	Synchronization States
$0.000 \sim 0.450$	Asynchronous
$0.450 \sim 0.500$	Anti-phase Synchronization and Asynchronous
$0.500 \sim 0.536$	In-phase Synchronization and Asynchronous
$0.536 \sim 1.000$	Synchronization with a certain phase

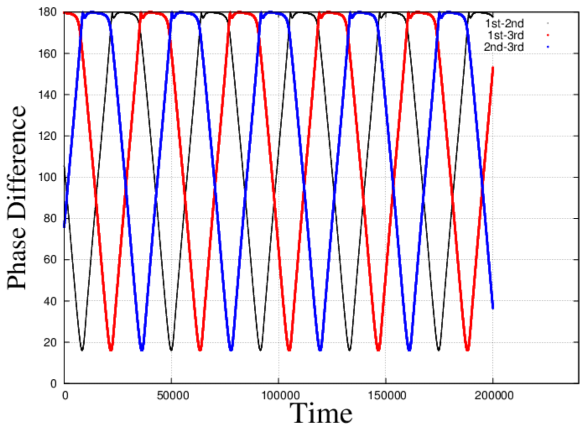

Fig. 10. Phase difference between adjacent shared triangular oscillators.

First, the case of β is less than 0.450, we can find that the phase differences between adjacent oscillators change periodically. Then, the bigger value of β, the width of change of the phase difference becomes larger and the case of β is over 0.450, the synchronization states shift the switching of anti-phase synchronization and asynchronous. Figure 10 shows the phase differences between shared three coupled oscillators about the case of β = 0.480. Each synchronization state is switching anti-phase synchronization and asynchronous, and three changes of phase differences remain the timing and shift. Moreover, sojourn time of anti-phase synchronization depend on the coupling strength and it tends to lengthen the coupling strength bigger.

If β reaches 0.500, we can find maintenance of in-phase synchronization, and the case of β is over 0.530, the in-phase synchronization states are maintained than anti-phase states. Then, the phase differences temporarily maintain about 160° in anti-phase area. The case of β is over 0.5350, the phase differences stay at in-phase in

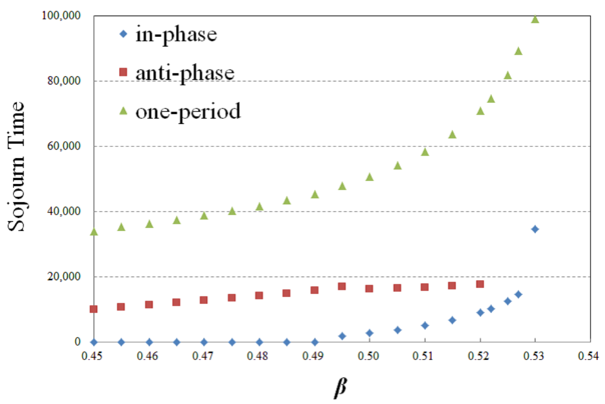

Fig. 11. Change of sojourn time by changing β.

regular intervals and they change the complex in anti-phase area. Figure 11 shows the sojourn time of one period, in-phase state and anti-phase. In this figure, we define the in-phase and anti-phase area from 0° to 10° and from 170° to 180°. Finally, β is over 0.5360, each phase difference converges a certain value. Namely, the phase differences change by γ, but they do not change qualitatively.

V. Conclusions

In this study, we have investigated synchronization phenomena observed in coupled two tetrahedrally form sharing the triangular oscillatory system. Unlike the case only for the tetrahedron form, it was observed that the phase difference converged with certain values. Especially, the coupling strength grew big, we could find the synchronization with in-phase between shared oscillators and synchronization with anti-phase between not shared oscillators. Also, we could find that the switching of asynchronous and anti-phase states when the coupling strength differ between shared oscillators and other one.

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