Foliated bifurcation structure in a 2-D coupled logistic map

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***Abstract*—In this study, we investigate bifurcations in coupled logistic maps whose parameters are forced into periodic varying in two-dimensional case. On a parameter plane, crossroad areas centered at fold cusp points regarding to several orders are detected. Especially, we investigate bifurcation curves regarding order 2 periodic orbits in detail. From the investigation, a foliated bifurcation structure is drawn, and existence domains of stable order 2 cycles with synchronization or without synchronization are detected. Moreover, evolution of bifurcation curves with respect to a coupling intensity is analyzed.**

I. Introduction

The study of information processing in brain using nonlinear models has attracted extensive attention during the last years. This attention is based on the consequence that neuron activity can be described by deterministic dynamical nonlinear equations. Indeed, studies by using equipments which record global brain activity like electroencephalogram and thermogram are carried out. However, there are problems depending on capabilities of the equipments. Therefore, we come up with the way to solve the problem by using model of brain with nonlinear systems. It is rich worth to introduce models to investigate nonlinear neuronal activities and mutual influences between neurons in the neural networks by using nonlinear maps in order to understand brain activities. It is known that if an autonomic neuron is influenced by external force, its characteristics are changed such as the neuron behaves periodic or chaotic [1]. How to give periodic external force is thought about in various ways, but we focus on a system whose parameter is forced into periodic varying and investigate behavior of the system and its pint-size coupling. The fundamental researches of these small coupled nonlinear systems are important for the investigation of global brain activities and the modeling, since we can think that general properties of large coupled nonlinear systems can de deduced from those of the its small size coupling system.

As the research for a system with parameter forced into periodic varying, [2] reports that the almost periodic oscillation occurs in a simple oscillator including parametric excitation if nonlinear inductor has saturation characteristic. Additionally the occurrence of chaos is referred in [3] and [4]. Previously, we have investigated synchronization phenomena in coupled logistic maps involving parametric force [5].

This study also deals with the coupled logistic maps whose parameters are forced into periodic varying. We investigate bifurcations in two-dimensional cases. As previously said, this low dimension can give rise to specific phenomena that we can explain and that are basic phenomena for the behavior of higher size neural networks. From the obtained bifurcations, foliated bifurcation structure is drawn, and existence domain of stable order 2 cycles with synchronization or without synchronization are detected. Moreover, evolution of bifurcation curves with respect to a coupling intensity is investigated. The bifurcation curves which appear in coupled case spread by increasing of the coupling intensity. The evolution with respect to the coupling is analytically proved.

II. System

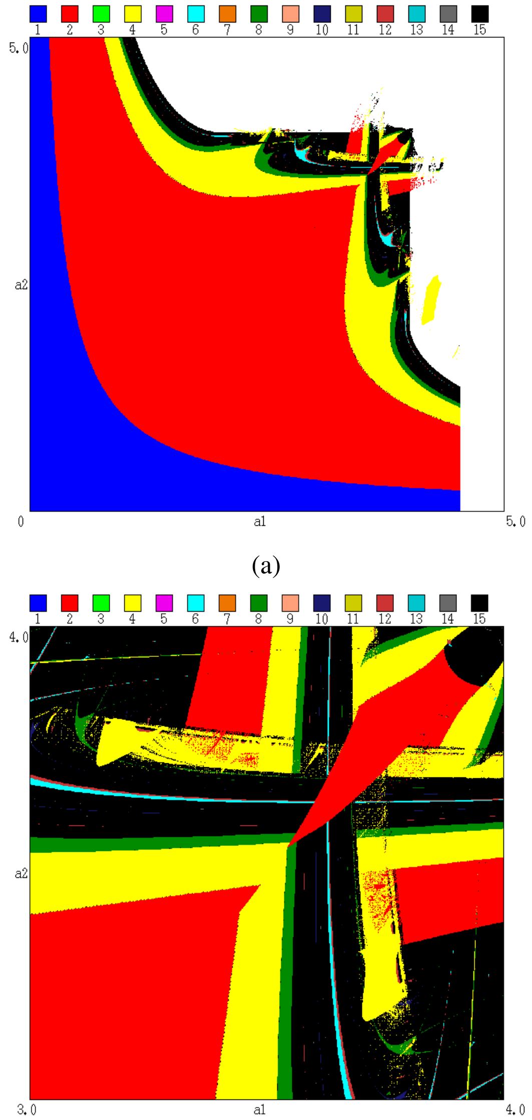
This paper treats a two-dimensional system. This system is obtained from the coupling with mutually influencing scheme of the same one-dimensional system whose parameters are forced into periodic varying. The system is described as:

$$T: \begin{cases} x(n+1) = (1-\frac{\varepsilon}{2})F(x(n),\alpha_f(n)) + \frac{\varepsilon}{2}F(y(n),\alpha_f(n)) \\ y(n+1) = (1-\frac{\varepsilon}{2})F(y(n),\alpha_f(n)) + \frac{\varepsilon}{2}F(x(n),\alpha_f(n)) \\ \qquad (n=1,2,...) \end{cases}, \tag{1}$$

and

$$\alpha_f(n) = \begin{cases} \alpha_1, & \text{for each even value of "}n\text{"} \\ \alpha_2, & \text{for each odd value of "}n\text{"} \\ & (n=1,2,...) \end{cases}. \tag{2}$$

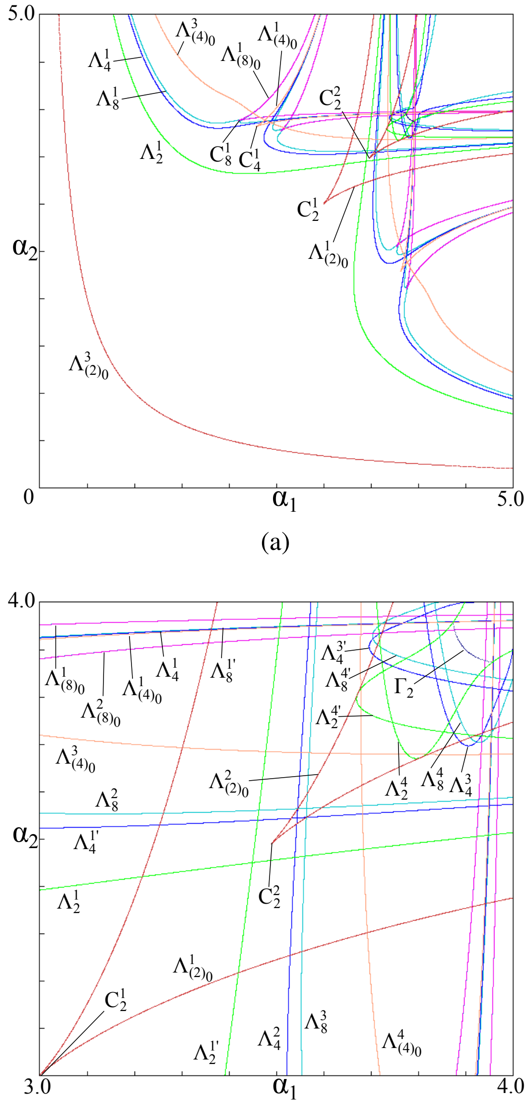
The parameter α_f alternately changes from α_1 to α_2 and $\varepsilon \in (0,1)$. This system is not continuous in iteration because α_f is periodically changed. Whereas, the twice iteration of this system is considered as continuous system since its periodicity. This system has the above characteristics. From this characteristics, non-existence of odd order cycle except fixed point and special situation $\alpha_1 = \alpha_2$ are prouved in [6]. In this study, F is a one-dimensional logistic map and



(a)

(b)

Fig. 1. Representation of the parameter plane (α_1, α_2) for ε = 0.24. (b) is an enlargement of (a).



(a)

(b)

Fig. 2. Bifurcation curves of order 2, 4 and 8 cycles indicating only the fold and flip curves according to boundaries between existence domains of stable cycles.

is described as:

$$\begin{aligned} F(x(n), \alpha_f(n)) &= \alpha_f(n)x(n)(1 - x(n)) \\ F(y(n), \alpha_f(n)) &= \alpha_f(n)y(n)(1 - y(n)). \end{aligned} \tag{3}$$

III. Bifurcations

Here, we consider the two-dimensional case, two parametrically forced logistic maps are coupled.

Figure 1 shows bifurcation diagrams for $\varepsilon = 0.24$ and its enlargement. Each colored domain indicates existence of at least one stable cycle whose order corresponds to upper colored squares in the figure. Figure 2 shows bifurcation curves of order 2, 4 and 8 cycles indicating only the curves according to boundaries between existence domains of stable cycles. Bifurcation curves are obtained using analytical-numerical methods [7]. In the figure, $\Lambda^j_{(k)_0}$, $\bar{\Lambda}^j_{(k)_0}$, Λ^j_k and Γ_k denote fold, pitchfork, flip and Neïmark-Sacker bifurcation curves, respectively. The index j is a number characterizing the considered order k cycle. C^j_k denotes a cusp point of an order k cycle. The bifurcation curves well correspond to the boundaries between colored domains in Fig. 1. On the parameter plane, crossroad areas (in the sense of [8]) centered at fold cusp points regarding to several orders can be detected.

A. *Foliated bifurcation structure of order 2 cycle*

Here, bifurcations regarding order 2 cycle are considered. Figure 3 shows fold, flip, pitchfork and Neïmark-Sacker bifurcation curves. In the figure, arrowed lines correspond to cascades of order 2 cyclic points associated with fold and

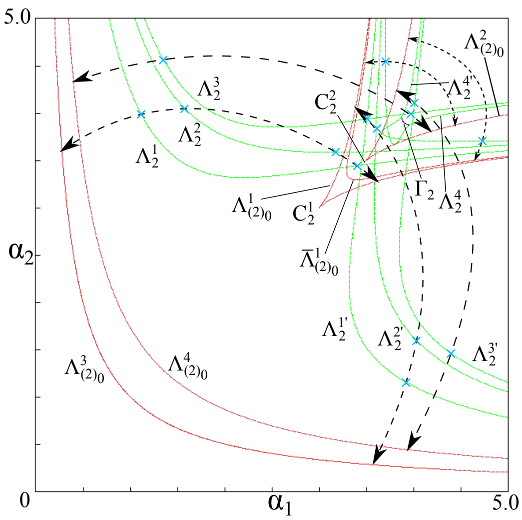


Fig. 3. Bifurcation curves of order 1 cycle of T^2_{odd} for ε = 0.24.

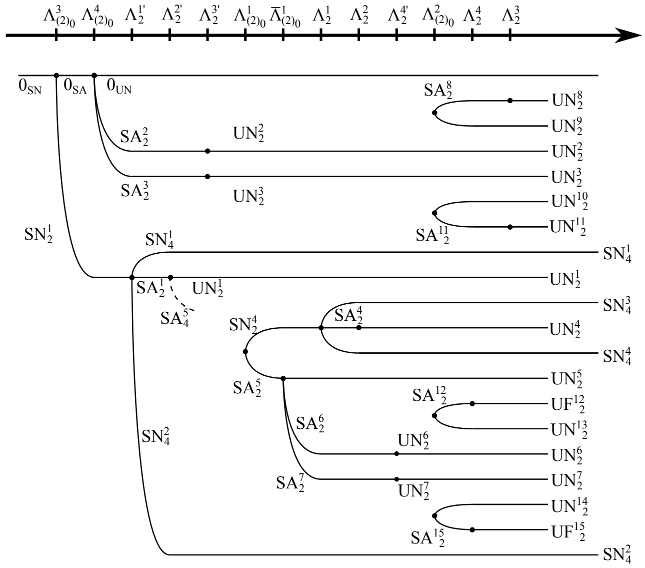


Fig. 4. Representation of bifurcations of order 2 cycles of T and its stabilities when α_1 is fixed and α_2 increases. We do not consider unstable order 4 cycle. SN^j_k, SA^j_k, UN^j_k and US^j_k indicate stable node, saddle, unstable node and unstable spiral of order k cyclic point, respectively. The index j characterizes the permutation of the order k cycle. 0 is fixed point at $(x, y) = (0, 0)$.

pitchfork bifurcations at which heads of the arrowed lines point. Christcrosses on the arrowed lines indicate that the order 2 cyclic points on the cascades undergo flip bifurcations. For instance, an order 2 cycle on the arrowed line starting from $\Lambda^3_{(2)_0}$ in left upper side of the figure undergoes flip bifurcations Λ^1_2 and Λ^2_2, and finally reaches to a fold bifurcation $\Lambda^1_{(2)_0}$ at which the head of the arrowed line points. The cycle on the cascade is not associated with other bifurcations like $\bar{\Lambda}^1_{(2)_0}$, Λ^3_2, $\Lambda^{1'}_2$, $\Lambda^{2'}_2$ and left side of $\Lambda^1_{(2)_0}$ through which the arrowed line passes. On the other hand, an order 2 cycle on the arrow line starting from $\Lambda^3_{(2)_0}$ in right lower side of the figure undergoes flip bifurcations $\Lambda^{1'}_2$ and $\Lambda^{2'}_2$, and finally reaches to a fold bifurcation $\Lambda^1_{(2)_0}$ at which the head of the arrowed line points. The cycle on the cascade is not associated with other bifurcations. From these relationships, $\Lambda^1_{(2)_0}$, $\Lambda^3_{(2)_0}$, Λ^1_2, Λ^2_2, $\Lambda^{1'}_2$ and $\Lambda^{2'}_2$ organize a crossroad area centered at a fold cusp C^1_2. In the same way, $\Lambda^2_{(2)_0}$, $\Lambda^4_{(2)_0}$, Λ^3_2 and $\Lambda^{3'}_2$ organize a crossroad area centered at a fold cusp C^2_2. $\Lambda^2_{(2)_0}$, $\bar{\Lambda}^1_{(2)_0}$, Λ^4_2, $\Lambda^{4'}_2$ organize a communication area including codimension-2 points. Here, it is detected that arrow lines starting from $\Lambda^4_{(2)_0}$ and $\bar{\Lambda}^1_{(2)_0}$ reach to $\Lambda^2_{(2)_0}$. This is because $\Lambda^2_{(2)_0}$ is associated to four cycles.

In order to understand relationships between bifurcation curves and appearance or disappearance of cycles and their stabilities, we consider a cascade when α_1 is fixed ($\alpha_1 = 4$) and α_2 is increasing. Relationships between bifurcation curves and appearance or disappearance of cycles and their stabilities on the cascade are represented in Fig. 4. In the figure, dots indicate that cycles are related to bifurcations denoted in the top of the figure. At the points, appearance or disappearance of cycles occur and their stability changes. Because we focus on order 2 cycle, we do not consider unstable order 4 cycles and indicate only appearance of stable order 4 cycles in the figure.

With increasing α_2, cycles appear through fold and pitchfork bifurcations, and the number of cycles finally becomes 15. It confirms that $\Lambda^2_{(2)_0}$ relates to four distinct cycles.

From the above results, a foliated bifurcation structure, three-dimensional representation, is drown up in Fig. 5 as introduced in [8]. The foliated bifurcation structure is organized by five organizations that each organization has a fold cusp and the organizations overlap. Organization 2 and organization 2' have the same structures. Organization 3 and organization 3' also have the same structures. Organization 3 and organization 3' appear from $\Lambda^2_{(2)_0}$ in the center sheet of organization 1 and construct shapes like pockets. In the figure, a blue-slushed domain corresponds to existence of a stable fixed point $(x, y) = (0, 0)$. Red-slushed domains correspond to existence of stable order 2 cycles. It can be seen that stable order 2 cycles exist in organization 1, organization 3 and organization 3'. All coordinates of order 2 cycles belonging to organization 1 correspond to $x = y$. Therefore, in the parameter domain according to red-slushed domains in the organization 1, synchronized order 2 cycles exist. On the other hand, all coordinates of order 2 cycles belonging to the other organizations correspond to $x \neq y$. Therefore, in the parameter domain according to red-slushed domains in the organization 3 and the organization 3', unsynchronized order 2 cycles exist.

B. Evolution of bifurcation curves for ε

Here, evolution of bifurcation curves with respect to third parameter ε is investigated. Figure 6 shows bifurcation curves with different values of ε. When ε is small, $\Lambda^4_{(2)_0}$ is close to $\Lambda^3_{(2)_0}$, $\Lambda^2_{(2)_0}$ and $\bar{\Lambda}^1_{(2)_0}$ are close to $\Lambda^1_{(2)_0}$, Λ^2_2 and Λ^3_2 are close to Λ^1_2, $\Lambda^{2'}_2$ and $\Lambda^{3'}_2$ are close to $\Lambda^{1'}_2$, Λ^4_2 and $\Lambda^{4'}_2$ are

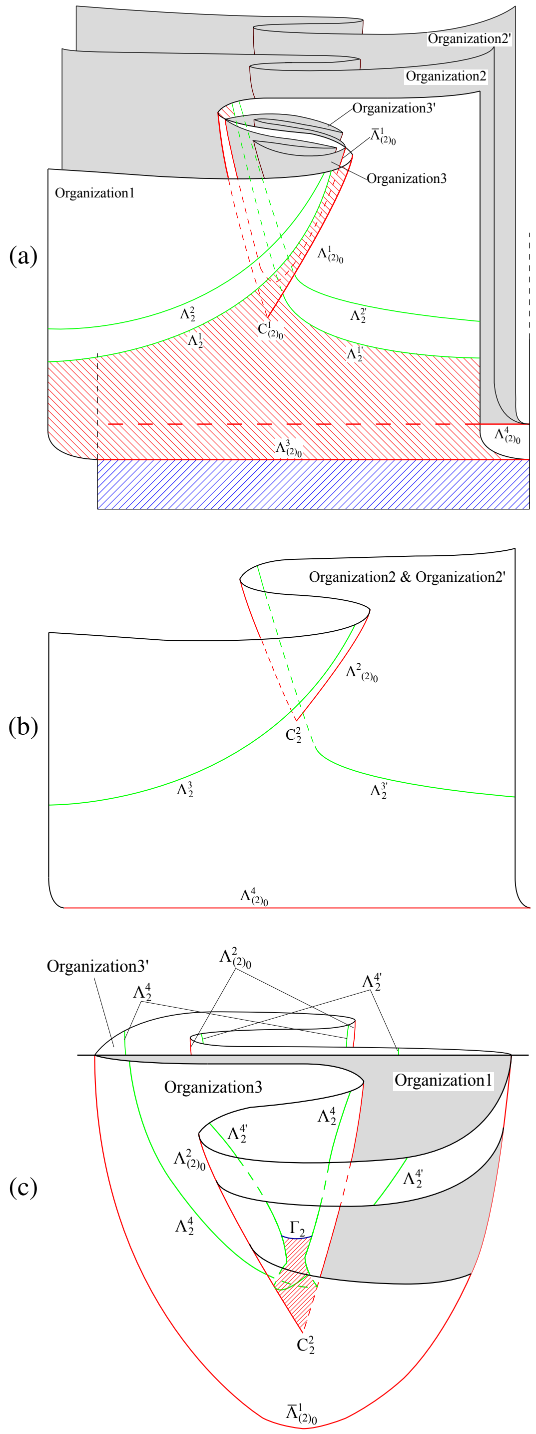


Fig. 5. Three dimensional representation of Fig. 3.

close to parts of Λ_2^1 and $\Lambda_2^{1'}$. With increasing ε, $\Lambda_{(2)_0}^2$, $\Lambda_{(2)_0}^4$, $\bar{\Lambda}_{(2)_0}^1$, Λ_2^2, $\Lambda_2^{2'}$, Λ_2^3, $\Lambda_2^{3'}$, Λ_2^4 and $\Lambda_2^{4'}$ spread and finally tend toward infinity when $\varepsilon = 1$, while $\Lambda_{(2)_0}^1$, $\Lambda_{(2)_0}^3$, Λ_2^1 and $\Lambda_2^{1'}$ do not move. Here, we prove these obtained results, analytically.

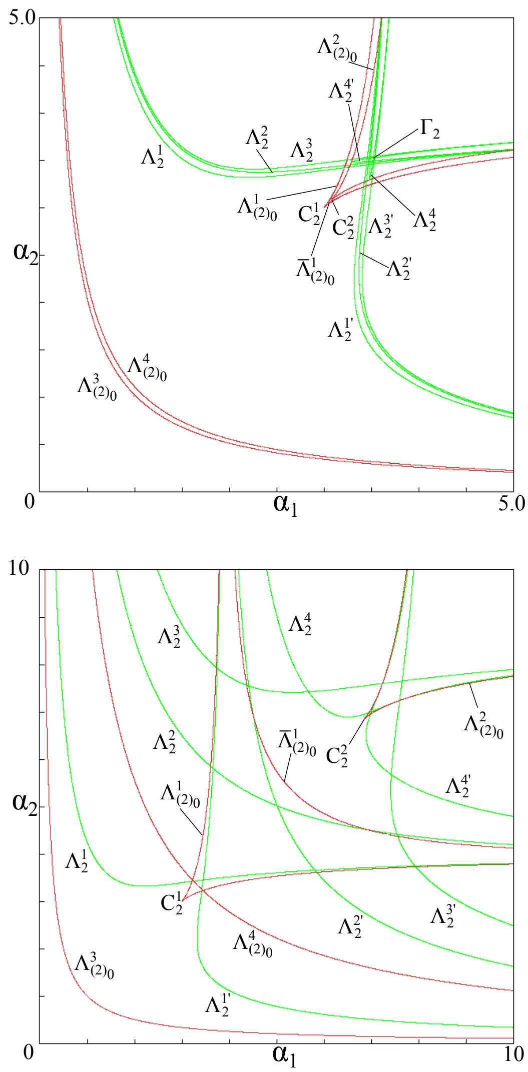


Fig. 6. Evolution of bifurcation curves with respect to third parameter ε. (a)$\varepsilon = 0.05$. $(b)\varepsilon = 0.70$.

$\Lambda_{(2)_0}^3$ and $\Lambda_{(2)_0}^4$ are described as

$$\begin{aligned} \Lambda_{(2)_0}^3 &: \alpha_1 \alpha_2 = 1 \\ \Lambda_{(2)_0}^4 &: \alpha_1 \alpha_2 = \frac{1}{1 - 2\varepsilon + \varepsilon^2} \end{aligned} \quad . \tag{4}$$

From these equations, $\Lambda_{(2)_0}^3$ is not related to ε and $\Lambda_{(2)_0}^4$ is related to ε and tends toward infinity at $\varepsilon = 1$. It is difficult to solve other bifurcation curves analytically. Then, we consider crossing points where fold and pitchfork bifurcation curves cross the diagonal of (α_1, α_2) parameter plane. We denote the crossing points where pitchfork bifurcation curves $\bar{\Lambda}_{(2)_0}^1$ cross the diagonal as $*\bar{\Lambda}^1_{(2)_0}$. The crossing points are described as

$$\begin{aligned} &C_2^1 : \alpha_1 = \alpha_2 = 3 \\ &C_2^2 : \alpha_1 = \alpha_2 = \frac{-1+\varepsilon-\sqrt{\varepsilon^2-2\varepsilon+4}}{-1+\varepsilon} \\ &*\bar{\Lambda}^1_{(2)_0} : \alpha_1 = \alpha_2 = 2 + \frac{1}{1-\varepsilon} \end{aligned} \quad . \qquad (5)$$

From these equations, C_2^1 is constant with respect to ε, and C_2^2 and $*\bar{\Lambda}^1_{(2)_0}$ relate to ε and tend toward infinity at $\varepsilon = 1$. Only the crossing points between bifurcation curves and the diagonal can be solved, analytically. However, the crossing points are parts of bifurcation curves and each bifurcation curve is continuous. Therefore, it is clear that the evolution of the crossing points is correlated with the evolution of their whole bifurcation curves.

IV. Conclusion

In this study, we investigated bifurcations in coupled logistic maps whose parameters are forced into periodic varying in two-dimensional case. From the investigation of bifurcation diagrams and bifurcation curves, existence domains of several stable cycles and chaotic orbits have been detected. The bifurcation curves construct crossroad areas centered at fold cusp points. Especially, we investigated bifurcation curves regarding to order 2 cycle in detail. From the investigation, a foliated bifurcation structure has been drawn, and existence of synchronized and non-synchronized cycles have been derived. Moreover, evolution of bifurcation curves with respect to a coupling intensity was investigated. The bifurcation curves caused by the coupling spread by increasing of the coupling intensity and tend toward infinity for the coupling intensity beeing equal to 1. The evolution with respect to the coupling has been analytically proved.

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