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Synchronization of Oscillators as Coupling Two Tetrahedrons with a Face

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Abstract—In this study, we investigate the synchronization phenomenon observed in two coupled tetrahedral oscillators sharing a face. We observe the several types of synchronization phenomena by changing the parameters. Furthermore, we investigate the effect of the initial conditions.

1. Introduction

Synchronization is one of the most typical natural phenomena, in particular, the cardiac heartbeat is well known as the example of the synchronization phenomena. Synchronization phenomena are one of the nonlinear phenomena, and they can be expressed by using coupled oscillators. Coupled oscillatory systems are suitable model to indicate high dimensional nonlinear phenomena in the natural science fields. Therefore investigations of the coupled oscillatory systems are reported in various research fields [1]-[7]. Various coupled oscillators were proposed, and the part of the mechanism of the non-linear phenomenon has been elucidated until now. We can see that coupled oscillatory systems produce interesting phase wave patterns, including wave propagation, clustering and complex patterns. However, synchronization phenomena of the oscillators have not been analyzed enough yet. Hence, we need to investigate the case of more complicated synchronization phenomena to explore high dimensional nonlinear phenomena.

Here, van der Pol oscillators were coupled in various form and were investigated about their synchronization phenomena [8], [9]. And we have studied three coupled oscillatory systems with a ring topology [12] as shown in Fig. 1(a). In this circuit system, each oscillator was coupled by an inductor and the number of coupled oscillators was an odd number. And then the coupled oscillators could not synchronize with in/anti-phase states. In other words, three-phase synchronization (phase shift: 120°) is obtained for the case of three oscillators by the effect of frustration. However, the three-phase synchronization was always observed stably in that system. In our previous study, we have investigated several kinds of interesting synchronization phenomena in coupled oscillatory system which has stronger frustrations. We have researched four coupled van der Pol oscillators in the regular tetrahedron form as shown in Fig. 1(b). By computer simulation, we found that the phase difference between adjacent oscillators changed and the synchronization was destroyed after the adjacent oscillators synchronize with anti-phase. In other words, this circuit model has the feature such as repeated anti-phase synchronous and asynchronous.

The another study, synchronization phenomena in two coupled triangular oscillatory networks sharing a branch was investigated in [13]. In this case, we could observed synchronization that the phase difference of sharing branch is in-phase (phase difference: 0°), and the other one between adjacent oscillators are synchronized with anti-phase (phase difference: 180°).

In this study, we investigate the synchronization phenomena of oscillatory model which is coupled two tetrahedrons with sharing a triangular face as shown in Fig. 2. In addition, we calculate the phase differences between the adjacent oscillators in coupled van der Pol oscillators.

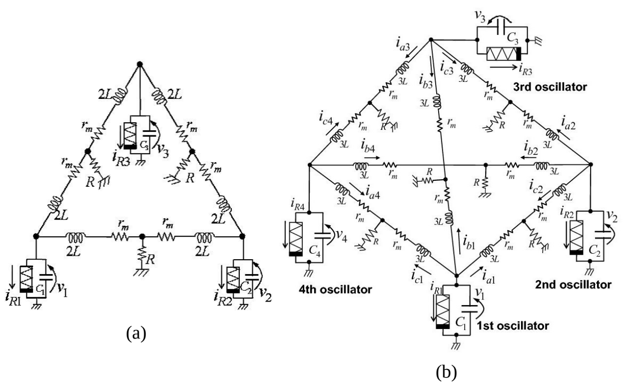


Figure 1: Coupled van der Pol oscillators. (a) Three coupled oscillator. (b) Four coupled oscillator.

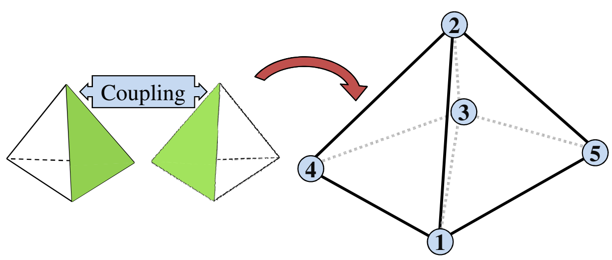
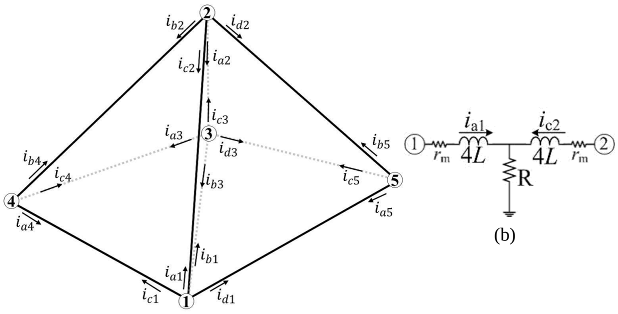


Figure 2: Coupled oscillator in tetrahedron form.



(a)

Figure 3: Circuit model with double tetrahedrons. (a) Conceptual circuit model. (b) Coupled structure.

2. Circuit Model

The circuit model is shown in Fig. 3(a). In this circuit model, two tetrahedrons oscillators are coupled by the triangular face and the fourth and the fifth oscillators have no connection. In the computer simulations, we assume that the $v_k - i_{Rk}$ characteristics of nonlinear resistor in each oscillator is given by the following third order polynomial equation.

$$i_{Rk} = -g_1 v_k + g_3 v_k 3 \qquad (g_1, g_3 > 0), \; (k = 1, 2, 3, 4, 5). \tag{1}$$

[First oscillator]

$$\begin{cases} \dfrac{dx_1}{d\tau} = \varepsilon(1 - x_1 2)x_1 - (y_{a1} + y_{b1} + y_{c1} + y_{d1}) \\ \dfrac{dy_{a1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{a1} - \gamma(y_{a1} + y_{c2})\Big\} \\ \dfrac{dy_{b1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{b1} - \gamma(y_{b1} + y_{b3})\Big\} \\ \dfrac{dy_{c1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{c1} - \gamma(y_{c1} + y_{a4})\Big\} \\ \dfrac{dy_{d1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{d1} - \gamma(y_{d1} + y_{a5})\Big\}. \end{cases} \tag{2}$$

[Second oscillator]

$$\begin{cases} \dfrac{dx_2}{d\tau} = \varepsilon(1 - x_2 2)x_k - (y_{a2} + y_{b2} + y_{c2} + y_{d2}) \\ \dfrac{dy_{a2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{a2} - \gamma(y_{a2} + y_{c3})\Big\} \\ \dfrac{dy_{b2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{b2} - \gamma(y_{b2} + y_{b4})\Big\} \\ \dfrac{dy_{c2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{c2} - \gamma(y_{c2} + y_{a1})\Big\} \\ \dfrac{dy_{d2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{d2} - \gamma(y_{d2} + y_{b5})\Big\}. \end{cases} \tag{3}$$

[Third oscillator]

$$\begin{cases} \dfrac{dx_3}{d\tau} = \varepsilon(1 - x_3 2)x_k - (y_{a3} + y_{b3} + y_{c3} + y_{d3}) \\ \dfrac{dy_{a3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{a3} - \gamma(y_{a3} + y_{c4})\Big\} \\ \dfrac{dy_{b3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{b3} - \gamma(y_{b3} + y_{b1})\Big\} \\ \dfrac{dy_{c3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{c3} - \gamma(y_{c3} + y_{a2})\Big\} \\ \dfrac{dy_{d3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{d3} - \gamma(y_{d3} + y_{c5})\Big\}. \end{cases} \tag{4}$$

[Fourth oscillator]

$$\begin{cases} \dfrac{dx_4}{d\tau} = \varepsilon(1 - x_4 2)x_k - (y_{a4} + y_{b4} + y_{c4} + y_{d4}) \\ \dfrac{dy_{a4}}{d\tau} = \dfrac{1}{4}\Big\{x_4 - \eta y_{a4} - \gamma(y_{a4} + y_{c1})\Big\} \\ \dfrac{dy_{b4}}{d\tau} = \dfrac{1}{4}\Big\{x_4 - \eta y_{b4} - \gamma(y_{b4} + y_{b2})\Big\} \\ \dfrac{dy_{c4}}{d\tau} = \dfrac{1}{4}\Big\{x_4 - \eta y_{c4} - \gamma(y_{c4} + y_{a3})\Big\} \\ \dfrac{dy_{d4}}{d\tau} = \dfrac{1}{4}\Big\{x_4 - \eta y_{d4}\Big\}. \end{cases} \tag{5}$$

[Fifth oscillator]

$$\begin{cases} \dfrac{dx_5}{d\tau} = \varepsilon(1 - x_5 2)x_k - (y_{a5} + y_{b5} + y_{c5} + y_{d5}) \\ \dfrac{dy_{a5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{a5} - \gamma(y_{a5} + y_{d1})\Big\} \\ \dfrac{dy_{b5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{b5} - \gamma(y_{b5} + y_{d2})\Big\} \\ \dfrac{dy_{c5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{c5} - \gamma(y_{c5} + y_{d3})\Big\} \\ \dfrac{dy_{d5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{d5}\Big\}. \end{cases} \tag{6}$$

We use the following normalizations:

$$t = \sqrt{LC}\tau, \; v_k = \sqrt{\frac{g_1}{g_3}}\, x_k, \; i_{ak} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{ak},$$

$$i_{bk} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{bk}, \; i_{ck} = \sqrt{\frac{g_1 C}{g_3 L}}\, y_{ck},$$

$$\varepsilon = g_1 \sqrt{\frac{L}{C}}, \; \gamma = R\sqrt{\frac{C}{L}}, \; \eta = r_m \sqrt{\frac{C}{L}},$$

(k=1, 2, 3,... 5),

where ε is the nonlinearity, γ is the coupling strength, and η indicates the resistive component. In the computer simulations, we calculate the phase differences between adjacent oscillators. Here r_m denotes the internal resistance of an inductor. We calculate Eqs. (2)-(6) using a fourth-order Runge-Kutta method with the step size $h = 0.002$ in this

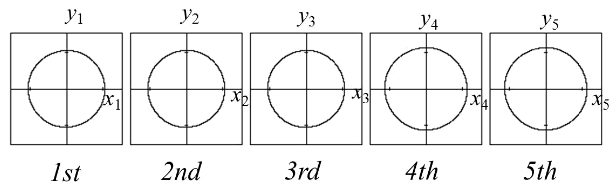


Figure 4: Attractor between adjacent oscillators (horizontal axis:y_k, vertical axis:x_k, (k = 1, 2, 3...5)).

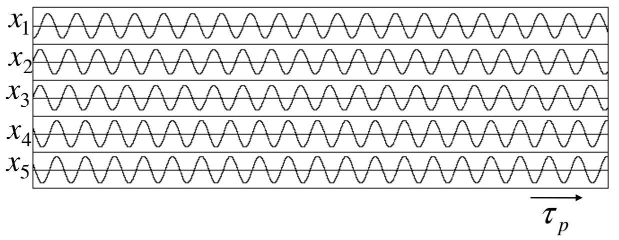


Figure 5: The time wave of each oscillator.

circuit. In this case, the parameters are set to $\varepsilon = 0.100$, $\eta = 0.00010$ and $\gamma = 0.100$. We measure the phase difference between adjacent oscillators with bisection method.

In Fig. 4, we show the attractor of each oscillator in this simulation. In this figure, we show the horizontal axis is the voltage of each condenser, and the vertical axis is the electric current of each oscillator. The electric current is summed the three currents $y_k = y_{ak} + y_{bk} + y_{ck} + y_{dk}$. And Fig. 5 shows the time wave form of the voltage of each capacitor after sufficient time has elapsed. The phase differences between the adjacent oscillator of this case is equal to the result as shown in Fig. 6. As a result, in the case of this circuit model, it was observed that the phase difference finally converged with the constant value. Additionally, we can find that synchronization with in-phase occurs in the sharing triangular oscillatory network. By changing the initial conditions, we could observe that either two adjacent oscillators synchronize with in-phase among sharing three oscillators (see. Fig. 7). This is one of the result when

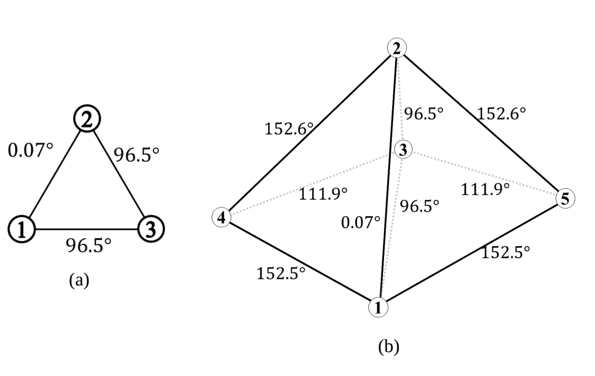


Figure 6: The phase difference between adjacent oscillator. (a) Sharing triangle. (b) The whole circuit model.

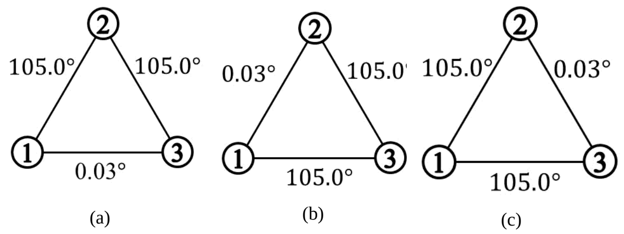


Figure 7: The relationship of phase differences in the sharing triangle for $\varepsilon = 0.50$, $\eta = 0.00010$, $\gamma = 0.10$.

we gave the different initial conditions by each simulation. By this result, we can find that the combinations of oscillators which synchronize with in-phase states change by given initial conditions. Also, we could not obtain three-phase synchronization in the sharing triangle.

3. Dependency of Initial Condition

In this section, we investigate the phase difference by changing the initial conditions. Except for the initial conditions of the first and second oscillator, we fix all parameters as follows, $\varepsilon = 0.10$, $\eta = 0.00010$, $\gamma = 0.10$. And all of initial conditions of the 3rd, 4th, and 5th oscillators is fixed as 0.10 (i.e. $x_k = y_{ak} = y_{bk} = y_{ck} = y_{dk} = 0.10$ ($k = 3, 4, 5$)) and iteration count is set to $\tau_p = 30,000$. In Fig.8(a), we map the phase difference between 1st and 2nd oscillators by changing initial conditions from 0.10 to 1.00. From this result, we confirm that the phase difference is divided into a few patterns. However, we find that a phase difference only rotates by initial conditions when we measured the phase difference between first oscillator and other one. Seen from another viewpoint, when we give the initial conditions of the two oscillators near the boundary, the phase difference changes from 0.0° to 94.5° by increasing the iteration. At the other case, the phase difference converges at the early stage far from the boundary.

In the case of $\varepsilon = 0.50$ and 1.00, we perform the above-mentioned measurement equally and the obtained results are summarized in Figs. 8 (b) and (c). By changing the nonlinearity (parameter ε), we can observe the rotation of the phase difference. Namely, the same combination of the phase difference is obtained. For example, we set the parameters as follows, $\varepsilon = 0.50$, initial condition of the 1st oscillator is 0.40 and the 2nd oscillator is 0.10. Then, we observe that the phase difference between 1st and 2nd oscillator is 109.5°, and 2nd and third oscillator is 0.03°.

In the case of the $\varepsilon = 1.00$, the combination of the phase differences changes with initial conditions likewise (see. Tab. 1). From this result, we can see that any two oscillators in triangular network are synchronized at in-phase state.

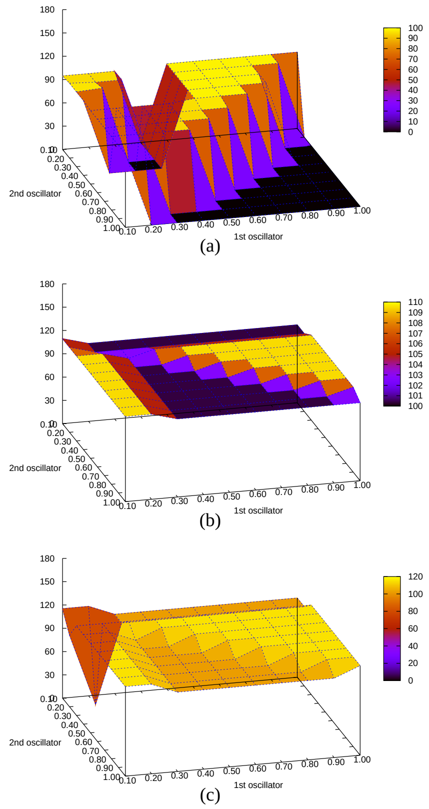


Figure 8: The phase differences between 1st and 2nd oscillators. (a) $\varepsilon = 0.10$. (b) $\varepsilon = 0.50$. (c) $\varepsilon = 1.00$.

4. Conclusion

In this study, we have investigated synchronization phenomena observed in coupled two tetrahedrally form sharing the triangle oscillatory system. Unlike the case only for the tetrahedron form, it was observed that the phase difference converged with a certain value. Furthermore, we made clear the characteristics of the phase differences by the initial conditions.

In our future works, we confirm that the stability of the oscillatory network which the coexistence of the solution may exist with circuit experiment and numerical analysis.

Table 1: Phase difference relations between the adjacent oscillators

Initial Conditions	Phase Difference			
for 1st-2nd	2nd	3rd	4th	5th
0.20-0.10	92.4°	92.4°	116.8°	116.8°
0.20-0.20	0.00°	102.1°	153.7°	153.7°
0.20-0.40	102.1°	0.00°	153.7°	153.7°
0.40-0.20	115.3°	115.3°	85.8°	85.8°

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