Spice-oriented Algorithm for Analysis of Coupled Oscillators

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***Abstract*—For designing oscillatory circuit, it is important to simulate and search the characteristics. If we analyze a weakly nonlinear circuit, we often apply averaging method which is one of the approximate solving method. In this study, we combine the averaging method to Newton homotopy method and analyze a circuit by using Spice. By solving Newton homotopy method, we obtain multiple equilibrium points in a single Spice simulation. As an example, we analyze the property of two coupled van der Pol oscillators. By using our proposed Spice-oriented algorithm, we obtain three equilibrium points. The result shows us, our proposed method is convenient for search the equilibrium points in averaging method.**

I. Introduction

The analysis of synchronization in coupled oscillators is very important in order to clarify mechanisms of the generations of various phenomena in natural systems. For example, blinking of the firefly, the moving rhythmically of heart cell, and laser oscillation and so on. In the field of electrical engineering, a lot of studies on synchronization phenomena of coupled van der Pol oscillators have been carried out up to now [1]-[3]. When each van der Pol oscillator produces a nearly sinusoidal wave shape, i.e., nonlinearity of the network is small, equilibrium points can be solved by some applying approximate methods such as averaging method [4], perturbation method, asymptotic method and so on. In generally, averaging method is widely used for analysis of weakly nonlinear systems in stable state. As particularly nice examples, Endo et al. have presented the details of a theoretical analysis by using averaging method and corresponding circuit experiments on coupled van der Pol oscillators arranged in a ladder, a ring and in a two-dimensional array topologies [5]-[7]. However, the number of variables and equilibrium points will be too large, by increasing the number of coupled oscillators. In this case, it is not easy to find all of equilibrium points.

In this study, we propose a convenient algorithm of averaging method combined with Newton homotopy method by using Spice (is application for circuit simulation, and many Spice-oriented algorithms are proposed by authors [8]-[10]). Usually, the circuit equation applied averaging method is including fixed integration. We approximate the fixed integration to trapezoidal formula in order to realize determining equation to dc circuit model. Newton homotopy method shows global convergent compared with Newton method, and we can obtain multiple equilibrium points. This method is realized in easily by using solution-curve tracing circuit (STC) in Spice [8]-[10]. By combining averaging method and Newton homotopy method, we obtain multiple equilibrium points with a single Spice simulation in parallel.

Section II describes averaging method. Section III explains Spice-oriented Newton homotopy method and STC. Illustrated examples of the proposed algorithm for the averaging method is shown in Sec. IV and Sec. V concludes the article.

II. Averaging method

Averaging method is used for solving the weakly nonlinear circuit systems. All of oscillatory circuits can be expressed by second order differential equation as Eq. (1).

$$\ddot{x} + x = \varepsilon f(t, x, \dot{x}). \tag{1}$$

$f(t, x, \dot{x})$ is a nonlinear function and ε satisfies $0 < \varepsilon \ll 1$ in Eq. (1). Equation (1) is given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = \varepsilon f(t, x, \dot{x}) - x \end{cases}. \tag{2}$$

If the nonlinearity of the circuit system is set to $\varepsilon = 0$ (this is because ε denotes the tiny constant), Eq. (2) can be described as Eq. (3).

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}. \tag{3}$$

Equation (3) satisfies

$$\begin{cases} x = \rho(t)\sin(t + \theta(t)) \\ y = \rho(t)\cos(t + \theta(t)) \end{cases}. \tag{4}$$

In Eq. (4), $\rho(t)$ and $\theta(t)$ are obtained by initial condition. From this reason, if ε is very small, we can express Eq. (2) as

$$\begin{cases} \dot{\rho}(t)\sin(t + \theta(t)) + \rho(t)(1 + \dot{\theta}(t))\cos(t + \theta(t)) \\ \quad = \rho(t)\cos(t + \theta(t)) \\ \dot{\rho}(t)\cos(t + \theta(t)) - \rho(t)(1 + \dot{\theta}(t))\sin(t + \theta(t)) \\ \quad = \varepsilon f(t, \rho(t)\sin(t + \theta(t)), \rho(t)\cos(t + \theta(t))) \\ \qquad -\rho(t)\cos(t + \theta(t)) \end{cases}.$$

Namely,

$$\begin{cases} \dot{\rho}(t) = \varepsilon f(t, \rho(t)\sin(t + \theta(t)), \rho(t)\cos(t + \theta(t))) \\ \qquad \cdot \cos(t + \theta(t)) \\ \dot{\theta}(t) = -\dfrac{\varepsilon}{\rho} f(t, \rho(t)\sin(t + \theta(t)), \rho(t)\cos(t + \theta(t))) \\ \qquad \cdot \sin(t + \theta(t)) \end{cases}. \tag{5}$$

In the averaging method, we can treat $\rho(t)$ and $\theta(t)$ to fixed numbers of ρ and θ. We can approximate $\rho(t)$ and $\theta(t)$ to average of $t = 0$ from 2π. We obtain

$$\begin{cases} \dot{\rho}(t) = \dfrac{\varepsilon}{2\pi}\displaystyle\int_0^{2\pi} f(\phi-\theta, \rho\sin\phi, \rho\cos\phi)\cdot\cos\phi d\phi \\ \dot{\theta}(t) = -\dfrac{\varepsilon}{2\pi\rho}\displaystyle\int_0^{2\pi} f(\phi-\theta, \rho\sin\phi, \rho\cos\phi)\cdot\sin\phi d\phi \\ \phi \equiv t+\theta \end{cases}. \quad (6)$$

If the steady state, $\dot{\rho}(t) = 0$ and $\dot{\theta}(t) = 0$ must be satisfied. From Eq. (6), we obtain a stable amplitude.

In our algorithm, we apply trapezoidal formula into integration in Eq. (6). Namely,

$$\begin{aligned} \dot{\rho}(t) &= \frac{\varepsilon}{2\pi}\int_0^{2\pi} f(\phi-\theta, \rho\sin\phi, \rho\cos\phi)\cdot\cos\phi d\phi \\ &= \frac{\varepsilon}{2\pi}(f(\theta,\rho,\phi_0) + f(\theta,\rho,\phi_K)) \\ &\quad +\frac{\varepsilon}{2\pi}(f(\theta,\rho,\phi_1)\cos\phi_1 + f(\theta,\rho,\phi_2)\cos\phi_2 \\ &\quad +\ldots + f(\theta,\rho,\phi_{K-1})\cos\phi_{K-1}) \end{aligned} \quad (7)$$

$$\begin{aligned} \dot{\theta}(t) &= -\frac{\varepsilon}{2\pi\rho}\int_0^{2\pi} f(\phi-\theta, \rho\sin\phi, \rho\cos\phi)\cdot\sin\phi d\phi \\ &= -\frac{\varepsilon}{2\pi\rho}(f(\theta,\rho,\phi_0) + f(\theta,\rho,\phi_K)) \\ &\quad -\frac{\varepsilon}{2\pi\rho}(f(\theta,\rho,\phi_1)\sin\phi_1 + f(\theta,\rho,\phi_2)\sin\phi_2 \\ &\quad +\ldots + f(\theta,\rho,\phi_{K-1})\sin\phi_{K-1}), \end{aligned} \quad (8)$$

where

$$\begin{aligned} &\phi_0 = 0,\ \phi_1 = \frac{1}{K}\cdot 2\pi,\ \phi_2 = \frac{2}{K}\cdot 2\pi, \\ &\ldots,\ \phi_{K-1} = \frac{K-1}{K}\cdot 2\pi,\ \phi_K = 2\pi. \end{aligned} \quad (9)$$

By exchanging Eq. (6) to Eqs. (7) and (8), we can express these equations by the dc circuits. $f(\theta,\rho,\phi_K)$ is realized by analog behavior models (ABMs) in Spice. ABM is the Spice-oriented function for realize an equation in Spice.

III. Spice-oriented Newton homotopy method

Newton homotopy method is one of method for finding multiple dc solutions. The circuit model of Newton homotopy method is shown in Fig. 1. We assume equations as follows;

$$\begin{cases} g_0(V_0, V_1, V_2, \ldots, V_M) = 0 \\ g_1(V_0, V_1, V_2, \ldots, V_M) = 0 \\ g_2(V_0, V_1, V_2, \ldots, V_M) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ g_{M-1}(V_0, V_1, V_2, \ldots, V_M) = 0 \\ g_M(V_0, V_1, V_2, \ldots, V_M) = 0 \end{cases}. \quad (10)$$

These determining equations are described by a set of algebraic equations, which consists of M-equations and same number of variables. However, it is not easy to solve the equations, because they may have the multiple solutions.

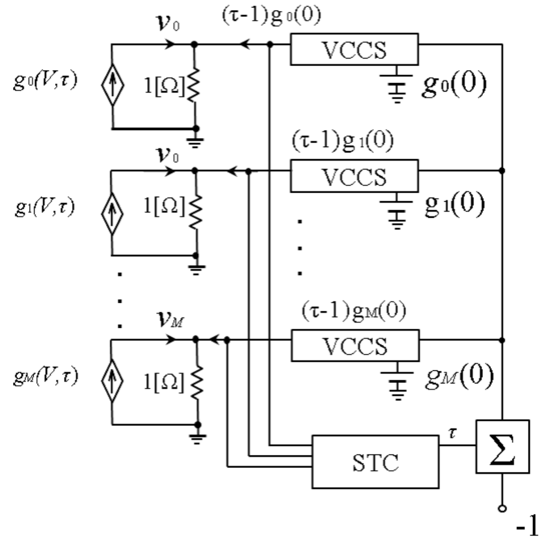

Fig. 1. Circuit model of Newton homotopy method.

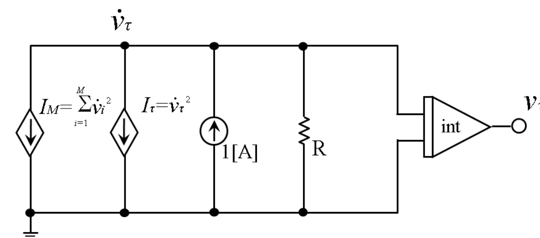

Fig. 2. Solution-curve tracing circuit (STC).

Applying the Newton homotopy method to solve Eq. (10), we obtain the following relation;

$$\boldsymbol{G}(\boldsymbol{V},\tau) = \boldsymbol{g}(\boldsymbol{V}) - (1-\tau)\boldsymbol{g}(\boldsymbol{V}_{(0)}) = \boldsymbol{0}. \quad (11)$$

where the initial state is set by a point $(\boldsymbol{V}_{(0)}, \tau = 0)$ and gets the solutions satisfying $g(v) = 0$ at $\tau = 1$ on the path. τ shows solutions curves called homotopy paths, and find the multiple solutions lying on the paths. A solution curve is traced by ark-length method as follows;

$$\begin{cases} \boldsymbol{G}(\boldsymbol{V},\tau) = \boldsymbol{0} \\ \displaystyle\sum^{M}\left(\frac{dv_i}{ds}\right)^2 + \left(\frac{d\tau}{ds}\right)^2 = 1 \\ i = 1 \\ i \neq 2 \end{cases}. \quad (12)$$

Equation (12) is realized by circuit model. Figure 2 shows the circuit diagram of solution-curve tracing circuit (STC).

IV. Illustrative example

A. Simulation circuit

As an illustrative example, we consider multi coupled oscillators as shown in Fig. 3. In this circuit model, two van

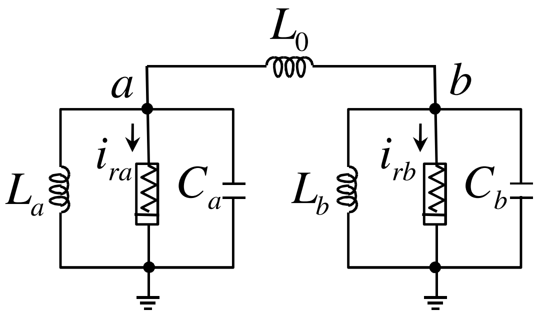

Fig. 3. Two van der Pol oscillators coupled by a inductor.

der Pol oscillators are coupled by a inductor. We can express the determining equation of a and b in Fig. 3 as follows;

$$\begin{cases} \dfrac{1}{L_a}\displaystyle\int v_a dt + C_a \dfrac{dv_a}{dt} + i_{ra} = \dfrac{1}{L_0}\displaystyle\int (v_a - v_b)dt \\ \dfrac{1}{L_b}\displaystyle\int v_b dt + C_b \dfrac{dv_b}{dt} + i_{rb} = \dfrac{1}{L_0}\displaystyle\int (v_b - v_a)dt \\ i_{ra} = -\alpha v_a + \beta {v_a}^3 \\ i_{rb} = -\alpha v_b + \beta {v_b}^3 \end{cases}. \tag{13}$$

We differentiate and arrange to Eq. (13).

$$\begin{cases} \dfrac{d^2 v_a}{dt^2} - \dfrac{\alpha}{C_a}\left(1 - \dfrac{3\beta}{\alpha}{v_a}^2\right)\dfrac{dv_a}{dt} \\ \qquad + \left(\dfrac{1}{C_a L_a} - \dfrac{1}{C_a L_0}\right)v_a + \dfrac{1}{C_a L_0}v_b = 0 \\ \dfrac{d^2 v_b}{dt^2} - \dfrac{\alpha}{C_b}\left(1 - \dfrac{3\beta}{\alpha}{v_b}^2\right)\dfrac{dv_b}{dt} \\ \qquad + \left(\dfrac{1}{C_b L_b} - \dfrac{1}{C_b L_0}\right)v_b + \dfrac{1}{C_b L_0}v_a = 0 \end{cases}. \tag{14}$$

We convert $(v_i,\ t)$ into

$$\begin{cases} v_i = \sqrt{\dfrac{\alpha}{3\beta}}x_i,\ \ i = a, b \\ t = \dfrac{t'}{\sqrt{\dfrac{1}{C_i L_i} - \dfrac{1}{C_i L_0}}} \end{cases}. \tag{15}$$

We obtain the normalized equations as follows;

$$\begin{cases} \ddot{x}_a - \varepsilon_a(1 - x_a^2)\dot{x}_a + x_a - \gamma_a x_b = 0 \\ \ddot{x}_b - \varepsilon_b(1 - x_b^2)\dot{x}_b + x_b - \gamma_b x_a = 0 \\ \varepsilon_i = \dfrac{\alpha}{\sqrt{\dfrac{1}{C_i L_i} - \dfrac{1}{C_i L_0}}},\ \ \gamma_i = \dfrac{L_i}{L_0 - L_i} \\ i = a, b \end{cases}. \tag{16}$$

In this study, we consider that two identical van der Pol oscillators are coupled. Namely, $\gamma_a = \gamma_b = \gamma$ and $\varepsilon_a = \varepsilon_b = \varepsilon$. We can express Eq. (16) as vector differential equation;

$$\ddot{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{x} = \varepsilon\dot{\boldsymbol{x}} - \frac{1}{3}\varepsilon\dot{\boldsymbol{x}}_c, \tag{17}$$

where

$$\boldsymbol{x} = [x_a, x_b]^T,\ \ \boldsymbol{x}_c = [{x_a}^3, {x_b}^3]^T, \boldsymbol{B} = \begin{bmatrix} 1 & -\gamma \\ -\gamma & 1 \end{bmatrix}. \tag{18}$$

We transform Eq. (18) to homogeneous as follows;

$$\begin{cases} \ddot{\boldsymbol{y}} + \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{y} = \varepsilon\dot{\boldsymbol{y}} - \dfrac{1}{3}\boldsymbol{P}^{-1}\varepsilon\dot{\boldsymbol{x}}_c \\ \boldsymbol{x} = \boldsymbol{P}\boldsymbol{y} \end{cases}. \tag{19}$$

In Eq. (19), eigenvalues of $\boldsymbol{B}$ are $\lambda_1 = 1 - \gamma$, $\lambda_2 = 1 + \gamma$. Although we can diagonalize $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$ by treating eigenvector of λ_1 is $p_1 = [1/\sqrt{2}, 1/\sqrt{2}]^T$ and λ_2 is $p_2 = [1/\sqrt{2}, -1/\sqrt{2}]^T$. Namely,

$$\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{P}^T\boldsymbol{B}\boldsymbol{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{20}$$

From this reason, we can express Eq. (19) as

$$\begin{cases} \ddot{y}_a - {\omega_a}^2 y_a = \varepsilon f_1(y_a, y_b, \dot{y}_a, \dot{y}_b) \\ \ddot{y}_b - {\omega_b}^2 y_b = \varepsilon f_2(y_a, y_b, \dot{y}_a, \dot{y}_b) \end{cases}, \tag{21}$$

where

$$\begin{cases} {\omega_a}^2 = \lambda_a, {\omega_b}^2 = \lambda_b \\ f_1(y_a, y_b, \dot{y}_a, \dot{y}_b) = \dot{y}_a - \dfrac{1}{3}g_1(y_a, y_b, \dot{y}_a, \dot{y}_b) \\ f_2(y_a, y_b, \dot{y}_a, \dot{y}_b) = \dot{y}_b - \dfrac{1}{3}g_2(y_a, y_b, \dot{y}_a, \dot{y}_b) \end{cases}. \tag{22}$$

In Eq. (22), g_1 and g_2 are given by

$$\begin{aligned} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T &= \boldsymbol{P}^{-1}\dot{\boldsymbol{x}}_c = \frac{d(\boldsymbol{P}^T\boldsymbol{x}_c)}{dt} \\ &= \frac{d}{dt}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \left(\frac{1}{\sqrt{2}}y_a + \frac{1}{\sqrt{2}}y_b\right)^3 \\ \left(\frac{1}{\sqrt{2}}y_a - \frac{1}{\sqrt{2}}y_b\right)^3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2}{y_a}^2\dot{y}_a + \frac{3}{2}\dot{y}_a{y_b}^2 + 3y_a y_b\dot{y}_b \\ \frac{3}{2}{y_b}^2\dot{y}_b + \frac{3}{2}\dot{y}_b{y_a}^2 + 3y_b y_a\dot{y}_a \end{bmatrix}. \end{aligned} \tag{23}$$

Namely,

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 - \frac{1}{2}{y_a}^2\dot{y}_a - \frac{1}{2}\dot{y}_a{y_b}^2 - y_a y_b\dot{y}_b \\ \dot{y}_2 - \frac{1}{2}{y_b}^2\dot{y}_b - \frac{1}{2}\dot{y}_b{y_a}^2 - y_b y_a\dot{y}_a \end{bmatrix}. \tag{24}$$

If ε is very small value ($\simeq 0$), we obtain the relation of y_a, $\dot{y}_a$, y_b and $\dot{y}_b$ as follows;

$$\begin{cases} y_a = \rho_a\sin(\omega_a t + \theta_a),\ \ \dot{y}_a = \rho_a\omega_a\cos(\omega_a t + \theta_a) \\ y_b = \rho_b\sin(\omega_b t + \theta_b),\ \ \dot{y}_b = \rho_b\omega_b\cos(\omega_b t + \theta_b) \end{cases}. \tag{25}$$

Equation (26) are given by applying an averaging method to Eq. (24).

$$\begin{cases} \dot{\rho}_i(t) = \dfrac{\varepsilon}{2\pi}\displaystyle\int_0^{2\pi}\dfrac{f(y_a, y_b, \dot{y}_a, \dot{y}_b)\cdot\cos(\omega_i t + \theta_i)}{\omega_i}d\phi \\ \dot{\theta}_i(t) = \dfrac{\varepsilon}{2\pi}\displaystyle\int_0^{2\pi}\dfrac{f(y_a, y_b, \dot{y}_a, \dot{y}_b)\cdot\sin(\omega_i t + \theta_i)}{\omega_i\rho_i}d\phi \\ i = a, b \end{cases}. \tag{26}$$

In our proposed algorithm, definite integration in Eq. (26) are realized by trapezoidal formula. We performed Newton homotopy method for STC combined with the circuit model of Eq. (26).

B. Simulation result

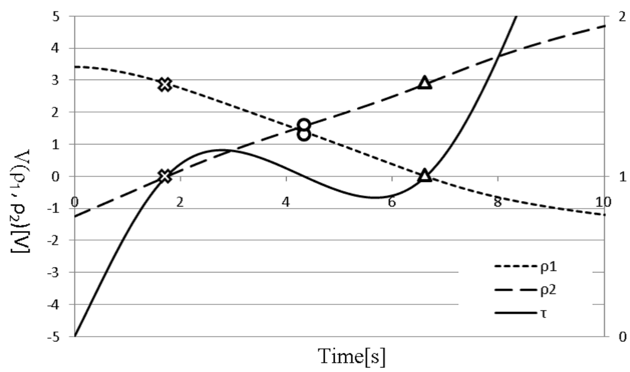

Fig. 4. Simulation result ($\gamma = 0.001$).

Figure 4 shows a simulation result of $\gamma = 0.001$ and $\varepsilon = 0.1$. We found three equilibrium points. Table I shows a solution which satisfies $\tau = 1$.

TABLE I
SOLUTIONS FOR $\gamma = 0.001$

time[s]	ρ_a	ρ_b
1.7359	2.8977	741.899μ
4.3208	1.4057	1.5719
6.6321	2.6675m	2.8953

TABLE II
EIGENVALUE OF SOLUTIONS

time[s]	λ_1	λ_2
1.7359	-17.2000	-8.7900
4.3208	-11.9584	5.7284
6.6321	-17.1001	-8.7699

Table II shows eigenvalue for each equilibrium point, which is obtained by calculation of MATLAB. Equilibrium points are asymptotic stability, when real part of λ satisfy < 0. Namely, we obtain asymptotic stably solutions where time = 1.7359 and 6.6321.

In this study, we set the relation of $\boldsymbol{x}$ and $\boldsymbol{y}$ as

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}, \boldsymbol{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \quad (27)$$

The stable equations are obtained by Eqs. (25) and (27).

$$x_a = x_b \simeq 2.05 \sin(\omega_a t + \theta_a) \quad (28)$$

where $\omega_a = \sqrt{1-\gamma} = 0.99949987$. Equation (28) shows in-phase solution.

$$x_a = -x_b \simeq 2.05 \sin(\omega_b t + \theta_b) \quad (29)$$

where $\omega_b = \sqrt{1+\gamma} = 1.00049988$. Equation (29) shows anti-phase solution.

From these results, we confirm that two steady states (in-phase and anti-phase states) coexist when the nonlinearity of the network is small.

In this example circuit, we find three equilibrium points in a transient analysis of Spice. We assume that more equilibrium points can be find in easily by using our proposed method, although the number of coupling oscillators becomes large.

V. CONCLUSION

We proposed Spice-oriented algorithm for averaging method to analyze the property of the coupled oscillatory systems. In our algorithm, we combine averaging method to Newton homotopy method. By introducing Newton homotopy method, we obtain multiple equilibrium points by the single Spice simulation. Where definite integration in averaging method is transformed into trapezoidal formula, and we realized it into dc circuit. As an example, we analyzed two van der Pol oscillators coupled by the inductor. By Spice simulation, we obtained three multiple equilibrium points.

In this study, we realized the circuit by normalized equation on Spice, and assess the stability with MATLAB. As a future works, we would like to expand an our proposed algorithm to perform an averaging method including assessment of stability by using only Spice.

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