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Synchronization in Coupled Triangular Oscillatory Networks Shared with Two Branches

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Abstract

In this study, we investigate phase difference in three coupled triangular oscillatory networks shared with two branches. We apply theoretical analysis considering the power consumption of coupling resistors to solve the phase difference. This theoretical analysis method matches well with the computer simulation results.

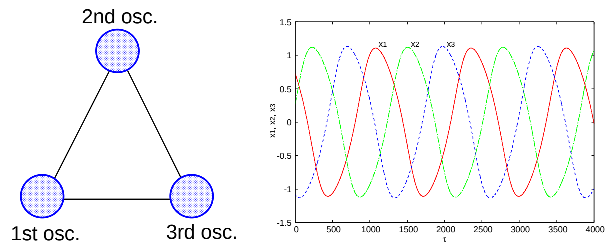
1. Introduction

Coupled oscillatory circuits provide simple models for describing high-dimensional nonlinear phenomena occurring in our everyday world. Synchronization phenomena have been extensively reported in physical, biological and electrical systems. Therefore, investigation of synchronization phenomena observed in the coupled oscillatory systems is important issue currently.

On the other hand, there are several types of polygonal network structures (e.g. Honeycomb structure and crystal structure) exists in the natural science. For the studies of large-scale network using coupled oscillators, a ring, a ladder and a two dimensional array structure are often investigated. However, there are not many discussions about coupled polygonal oscillatory networks by using electrical oscillators.

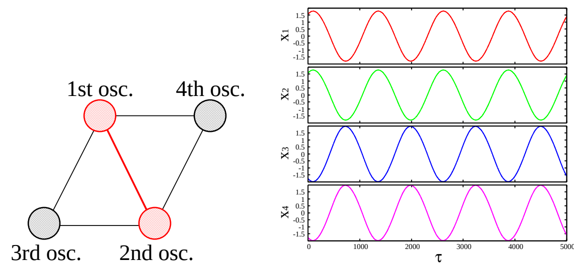
In general, when three van der Pol oscillators are coupled by resistor via inductor as ring topology, three phase synchronization (phase shift: $2\pi/3$) can be observed (see. Fig. 1). We assumed that some kinds of frustrations have ability to produce interesting synchronization phenomena. In our previous works, synchronization in two coupled triangular oscillatory networks sharing a branch (see. Fig. 2) was investigated [2]. By using computer simulations and theoretical analysis, we confirmed that the shared oscillators are synchronized at in-phase state. And the other combination oscillators synchronize with anti-phase state.

In this study, for understanding the general large system, synchronization phenomena in coupled triangular oscillatory networks sharing by two branches is investigated. By computer simulations, we confirm interesting synchronization which means that the adjacent oscillators are not synchronized with three-phase state. Furthermore, we apply theoretical analysis considering the power consumption for solving the phase difference.



(a) (b)

Figure 1: Three coupled van der Pol oscillators. (a) Circuit model. (b) Three phase synchronization.

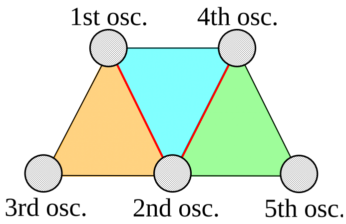


(a) (b)

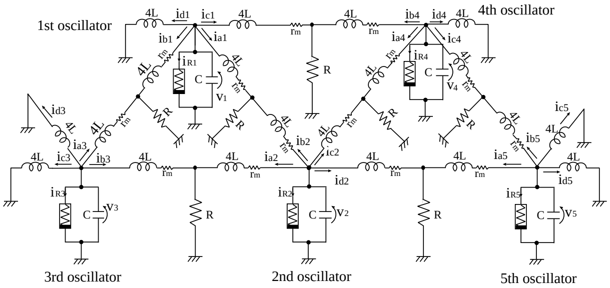
Figure 2: Two coupled triangular oscillatory networks. (a) Circuit model. (b) Synchronization state.

2. Circuit Model

The circuit model of this study is shown in Fig. 3. Three triangular oscillatory networks are coupled by sharing two branches. We use van der Pol oscillator as a coupled oscillator in this system. Each van der Pol oscillator is connected to every corners of the triangular networks.



(a) Conceptual circuit model.



(b) Circuit model.

Figure 3: Three coupled triangular oscillatory networks sharing with two branches.

The $v_k - i_{Rk}$ characteristics of the nonlinear resistor is approximated by the following third order polynomial equation,

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3 \quad (g_1, g_3 > 0), (k = 1, 2, 3, 4, 5). \tag{1}$$

The normalized circuit equations governing the circuit are expressed as

[1st oscillator]

$$\begin{cases} \dfrac{dx_1}{d\tau} = \varepsilon\Big(1 - \dfrac{1}{3}{x_1}^2\Big)x_1 - (y_{a1} + y_{b1} + y_{c1} + y_{d1}) \\ \dfrac{dy_{a1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{a1} - \gamma(y_{a1} + y_{b2})\Big\} \\ \dfrac{dy_{b1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{b1} - \gamma(y_{b1} + y_{a3})\Big\} \\ \dfrac{dy_{c1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{c1} - \gamma(y_{c1} + y_{b4})\Big\} \\ \dfrac{dy_{d1}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{d1}\Big\} \end{cases} \tag{2}$$

[2nd oscillator]

$$\begin{cases} \dfrac{dx_2}{d\tau} = \varepsilon\Big(1 - \dfrac{1}{3}{x_2}^2\Big)x_2 - (y_{a2} + y_{b2} + y_{c2} + y_{d2}) \\ \dfrac{dy_{a2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{a2} - \gamma(y_{a2} + y_{b3})\Big\} \\ \dfrac{dy_{b2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{b2} - \gamma(y_{b2} + y_{a1})\Big\} \\ \dfrac{dy_{c2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{c2} - \gamma(y_{c2} + y_{a4})\Big\} \\ \dfrac{dy_{d2}}{d\tau} = \dfrac{1}{4}\Big\{x_2 - \eta y_{d2} - \gamma(y_{d2} + y_{a5})\Big\} \end{cases} \tag{3}$$

[3rd oscillator]

$$\begin{cases} \dfrac{dx_3}{d\tau} = \varepsilon\Big(1 - \dfrac{1}{3}{x_3}^2\Big)x_3 - (y_{a3} + y_{b3} + y_{c3} + y_{d3}) \\ \dfrac{dy_{a3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{a3} - \gamma(y_{a3} + y_{b1})\Big\} \\ \dfrac{dy_{b3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{b3} - \gamma(y_{b3} + y_{a2})\Big\} \\ \dfrac{dy_{c3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{c3}\Big\} \\ \dfrac{dy_{d3}}{d\tau} = \dfrac{1}{4}\Big\{x_3 - \eta y_{d3}\Big\} \end{cases} \tag{4}$$

[4th oscillator]

$$\begin{cases} \dfrac{dx_4}{d\tau} = \varepsilon\Big(1 - \dfrac{1}{3}{x_4}^2\Big)x_4 - (y_{a4} + y_{b4} + y_{c4} + y_{d4}) \\ \dfrac{dy_{a4}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{a4} - \gamma(y_{a4} + y_{c2})\Big\} \\ \dfrac{dy_{b4}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{b4} - \gamma(y_{b4} + y_{c1})\Big\} \\ \dfrac{dy_{c4}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{c4} - \gamma(y_{c4} + y_{b5})\Big\} \\ \dfrac{dy_{d4}}{d\tau} = \dfrac{1}{4}\Big\{x_1 - \eta y_{d4}\Big\} \end{cases} \tag{5}$$

[5th oscillator]

$$\begin{cases} \dfrac{dx_5}{d\tau} = \varepsilon\Big(1 - \dfrac{1}{3}{x_5}^2\Big)x_5 - (y_{a5} + y_{b5} + y_{c5} + y_{d5}) \\ \dfrac{dy_{a5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{a5} - \gamma(y_{a5} + y_{d2})\Big\} \\ \dfrac{dy_{b5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{b5} - \gamma(y_{b5} + y_{c4})\Big\} \\ \dfrac{dy_{c5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{c5}\Big\} \\ \dfrac{dy_{d5}}{d\tau} = \dfrac{1}{4}\Big\{x_5 - \eta y_{d5}\Big\} \end{cases} \tag{6}$$

where

$$t = \sqrt{LC}\tau, \quad v_k = \sqrt{\frac{g_1}{3g_3}}x_k,$$

$$i_{ak} = \sqrt{\frac{g_1}{3g_3}}\sqrt{\frac{C}{L}}y_{ak}, \quad i_{bk} = \sqrt{\frac{g_1}{3g_3}}\sqrt{\frac{C}{L}}y_{bk},$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}}, \quad \gamma = R\sqrt{\frac{C}{L}}, \quad \eta = r_m\sqrt{\frac{C}{L}},$$

$$(k = 1, 2, 3, 4, 5).$$

In these equations, γ is the coupling strength, ε denotes the nonlinearity of the oscillators and η denotes the tiny resistor to avoid L-loop. For the computer simulations, these parameters are fixed as $\gamma = 0.1$, $\varepsilon = 0.1$ and $\eta = 0.0001$, respectively. By using Runge-Kutta method to solve the normalized circuit equation, we can simulate synchronization phenomena of this circuit system.

3. Synchronization Phenomena

3.1. Computer Simulations

First, the time wave forms obtained from each oscillator is shown in Fig. 4. Next, we calculate the phase difference between the adjacent oscillators. The simulated results is summarized in Tab. I. We confirm that coupled oscillators split up into four groups depended on phase difference (group-A: ≅96°, group-B: ≅133°, group-C: ≅130° and group-D: ≅167°). The coupled oscillators connected to the shared branches synchronize with ≅96° phase shift. Because, these oscillators have some kinds of dilemma.

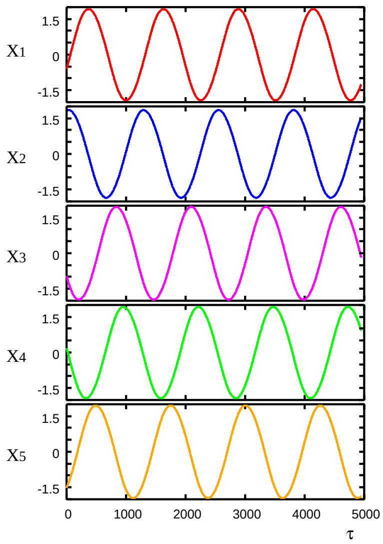


Figure 4: Time wave forms obtained from each oscillator.

Table 1: Synchronization phenomena

Combination	Phase difference
1st - 2nd	95.22°
2nd - 4th	97.05°
2nd - 3rd	133.26°
2nd - 5th	133.20°
1st - 3rd	131.42°
4th - 5th	129.75°
1st - 4th	167.69°

3.2. Theoretical Analysis

In our previous research, we have proposed the theoretical analysis to solve the phase difference of polygonal oscillatory networks [3]. We apply this theoretical approach to the circuit model of this study which is triangle oscillatory networks sharing with two branches.

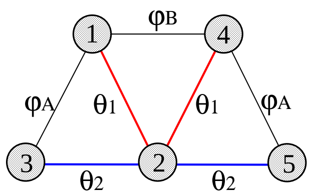


Figure 5: Definition the phase difference.

First, we define the two phase differences θ_1 and θ_2 as shown in Fig. 5. Then we can obtain the phase differences φ_A and φ_B as described by the following equations.

$$\begin{aligned} \varphi_A &= 2\pi - (\theta_1 + \theta_2), \\ \varphi_B &= 2\pi - 2\theta_1. \end{aligned} \tag{7}$$

The power consumption of the whole system is expressed by the following equations.

$$\begin{aligned} P &= \frac{2}{2\pi}\int_0^{2\pi}\{\sin\omega t + \sin(\omega t + \theta_1\}^2 dt \\ &+ \frac{2}{2\pi}\int_0^{2\pi}\{\sin\omega t + \sin(\omega t + \theta_2\}^2 dt \\ &+ \frac{2}{2\pi}\int_0^{2\pi}\{\sin\omega t + \sin(\omega t + 2\pi - (\theta_1 + \theta_2))\}^2 dt \\ &+ \frac{1}{2\pi}\int_0^{2\pi}\{\sin\omega t + \sin(\omega t + 2\pi - 2\theta_1)\}^2 dt, \end{aligned} \tag{8}$$

where, the amplitude of the current is set to 1 and the coupling resistance is fixed as R=1. By integrating Eq. (8), we obtain the equation of the power consumption.

$$\begin{aligned} P &= 2(1+\cos\theta_1) + 2(1+\cos\theta_2) \\ &+ 2(1+\cos(2\pi - (\theta_1+\theta_2)) \\ &+ 1 + \cos(2\pi - 2\theta_1). \end{aligned} \tag{9}$$

Figure 6 shows the function of power consumption with 3D plot when θ_1 and θ_2 are changed from $-\pi$ to π. By calculating this function, we obtain θ_1 and θ_2 which show the minimum value of this function. In the range of [0:π], the minimum value of θ_1 and θ_2 are 95° and 135°, respectively.

Table 2 summarizes the phase differences obtained from the theoretical analysis and the computer simulations using Eqs. (11), (12). We confirm that the phase differences of the theoretical analysis match pretty well to the results of the computer simulations. We consider that even if calculation becomes more complex for the asymmetrical system, the same theoretical analysis can be applied.

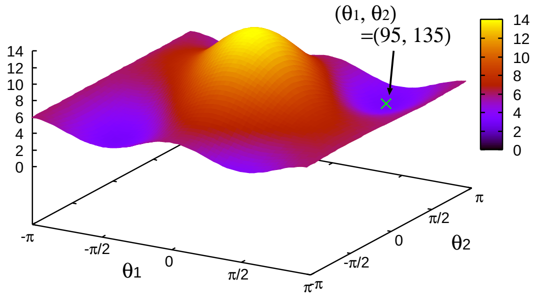


Figure 6: Power function of Eq. (9).

Table 2: Phase difference for oscillatory networks

Phase Type	Phase Difference	
	Theory	Simulation
θ_1	95.00°	95.22/97.05°
θ_2	135.00°	133.26/133.20°
φ_A	130.00°	131.42/129.75°
φ_B	170.00°	167.69°

4. Conclusions

In this study, we have investigated phase difference in three coupled triangular oscillatory networks shared with two branches. The interesting synchronization phenomena was observed. We assume that the adjacent oscillators are synchronized with some sort of rules.

Furthermore, we applied the theoretical analysis to solve the phase difference and confirm that this theoretical approach matched well with the computer simulation results.

Acknowledgment

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References

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