Penetration and Reflection Mechanisms of Phase-Inversion Waves in Lattice Oscillators

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***Abstract*—We observe and analyze particular synchronization phenomena on a lattice of coupled oscillators. Wave-motion phenomena which propagate phase differences can be observed on the system in steady states, when the system has nonlinearity. We call this wave-motion phenomena "phase-inversion waves." The phase-inversion waves have some behaviors, which are propagation, reflection, extinction, and penetration. In this paper, a penetration mechanism between a vertical phase-inversion wave and a horizontal phase-inversion wave is made clear. Furthermore, a reflection mechanism between two phase-inversion waves is made clear.**

I. Introduction

In this world, there are synchronization phenomena in every where[1]. The synchronization phenomena are existing among atoms, sea waves, planets and so on. There are synchronization phenomena also among humans as rumors, trends, and so on. Therefore, the synchronization phenomenon is researched in various fields. For example, there is research of synchronization phenomena of a lot of fireflies[2].We investigate and analyze the synchronization phenomena on the coupled oscillator systems[3]. In our previous study, we discovered the wave-motion which propagates phase difference, which is around 180 degrees, between adjacent oscillato rs on coupled oscillators system as a ladder and as a lattice. We call the wave-motion "phase-inversion wave." The phase-inversion waves can be observed in steady states[4]. There are the phase-inversion wave propagating to vertical direction and the phase-inversion wave propagating to horizontal direction on the lattice coupled oscillators system.

In this study, attracting forces to in-phase or anti-phase synchronization are investigated on one parameter-set where the phase-inversion waves can be observed. A penetration mechanism between the vertical phase-inversion wave and the horizontal phase-inversion wave and reflection mechanism between two phase-inversion waves are made clear by instantaneous frequencies of each oscillator, phase differences between adjacent oscillators, and the attracting forces.

II. Circuit model

A lot of van der Pol oscillators are coupled by inductors L_0 as a lattice(see Fig. 1). The number of column and the number of row of this system are assumed as "$M+1$" and "$N+1$." We name each oscillator OSC(k,l). A voltage of each oscillator is named $v_{(k,l)}$, and a current of inductor of each oscillator is named $i_{(k,l)}$(see Fig. 1). The circuit equations of this circuit model are normalized by Eq. (1), and the normalized circuit equations are shown as Eqs. (2)–(6).

$$\begin{aligned} &i_{(k,l)} = \sqrt{\frac{Cg_1}{3Lg_3}}x_{(k,l)}, \quad v_{(k,l)} = \sqrt{\frac{g_1}{3g_3}}y_{(k,l)}, \\ &t = \sqrt{LC}\tau, \quad \frac{d}{d\tau} = \text{“ } \cdot \text{ ”}, \alpha = \frac{L}{L_0}, \quad \varepsilon = g_1\sqrt{\frac{L}{C}}. \end{aligned} \tag{1}$$

[Corner–top] (left:(a,b)=$(0,1)$. right:(a,b)=$(N,N-1)$.)

$$\begin{aligned} \frac{dx_{(0,a)}}{d\tau} &= y_{(0,a)}, \\ \frac{dy_{(0,a)}}{d\tau} &= -x_{(0,a)} + \alpha(x_{(0,b)} + x_{(1,a)} - 2x_{(0,a)}) \\ &\quad + \varepsilon(y_{(0,a)} - \frac{1}{3}y^3_{(0,a)}). \end{aligned} \tag{2}$$

[Corner–bottom] (left:(a,b)=$(0,1)$. right:(a,b)=$(N,N-1)$.)

$$\begin{aligned} \frac{dx_{(M,a)}}{d\tau} &= y_{(M,a)}, \\ \frac{dy_{(M,a)}}{d\tau} &= -x_{(M,a)} + \alpha(x_{(M-1,a)} + x_{(M,b)} \\ &\quad - 2x_{(M,a)}) + \varepsilon(y_{(M,a)} - \tfrac{1}{3}y^3_{(M,a)}). \end{aligned} \tag{3}$$

[Center] ($0 < k < M$. $0 < l < N$.)

$$\begin{aligned} \frac{dx_{(k,l)}}{d\tau} &= y_{(k,l)}, \\ \frac{dy_{(k,l)}}{d\tau} &= -x_{(k,l)} + \alpha(x_{(k+1,l)} + x_{(k-1,l)} + x_{(k,l+1)} + x_{(k,l-1)} \\ &\quad - 4x_{(k,l)}) + \varepsilon(y_{(k,l)} - \tfrac{1}{3}y^3_{(k,l)}). \end{aligned} \tag{4}$$

[Edge]

(top:(a,b)=$(0,1)$.bottom:(a,b)=$(M,M-1)$.both:$0< l <N$.)

$$\begin{aligned} \frac{dx_{(a,l)}}{d\tau} &= y_{(a,l)}, \\ \frac{dy_{(a,l)}}{d\tau} &= -x_{(a,l)} + \alpha(x_{(a,l-1)} + x_{(a,l+1)} + x_{(b,l)} - 3x_{(a,l)}) \\ &\quad + \varepsilon(y_{(a,l)} - \tfrac{1}{3}y^3_{(a,l)}). \end{aligned} \tag{5}$$

(left:(a,b)=$(0,1)$. right:(a,b)=$(N,N-1)$. both:$0<k<M$.)

$$\begin{aligned} \frac{dx_{(k,a)}}{d\tau} &= y_{(k,a)}, \\ \frac{dy_{(k,a)}}{d\tau} &= -x_{(k,a)} + \alpha(x_{(k-1,a)} + x_{(k+1,a)} + x_{(k,b)} - 3x_{(k,a)}) \\ &\quad + \varepsilon(y_{(k,a)} - \tfrac{1}{3}y^3_{(k,a)}). \end{aligned} \tag{6}$$

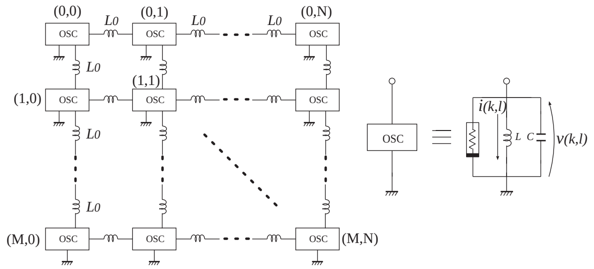

Fig. 1. Circuit Model.

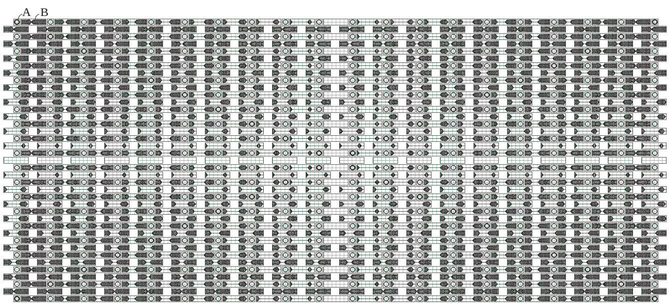

Fig. 2. The phase-inversion waves on 20x20 oscillators(α:an attractor of each oscillator(current vs. voltage), β:a sum of voltages of adjacent oscillators(sum of voltage vs. time)).

The α corresponds to the coupling parameter of each oscillator. The ε corresponds to the nonlinearity of each oscillator. This system is simulated by the fourth order Runge-Kutta methods using Eqs. (2)-(6). The phase-inversion waves are shown in Fig. 2. Figure 2–A expresses attractor of each oscillators. Figure 2–B expresses itinerancy of phase difference sum of voltages of adjacent oscillators is shown along the time.

III. Attracting Force

Attracting forces to in-phase or anti-phase synchronization are investigated on one parameter-set where the phase-inversion waves can be observed(see Fig.3). Attracting forces are observed as follows:

1) The phase differences between all adjacent oscillators are fixed as arbitrary value in the lattice system.
2) The phase difference between OSC(1,4) and OSC(1,5) after one period from initial value is analyzed along the initial phase difference is changed.

A vertical axis of Fig.3 expresses a variation of phase difference per one cycle. An upper direction shows attracting force to in-phase. A downward direction shows attracting force to anti-phase. A horizontal axis shows initial phase differences. Therefore, the length of line shows a attracting forces at each phase difference. Attracting force to in-phase is the strongest in 40 degrees. Attracting force to anti-phase is the strongest in 130 degrees.

IV. Behavior of Phase-Inversion Waves

We can observed some behaviors of phase-inversion waves on above systems. These behaviors are a propagation, a reflection at an edge, a reflection between phase-inversion waves, distinction and a penetration. Moreover, these behaviors can be classified by frequencies, because three frequencies are observed in steady states. The synchronizations for vertical

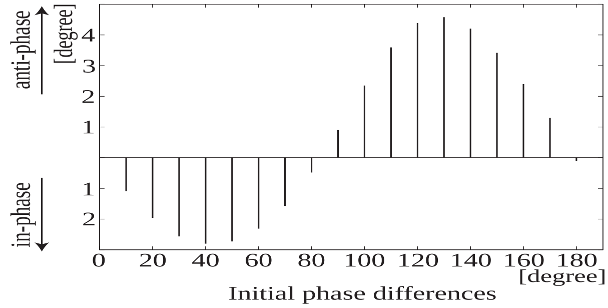

Fig. 3. Attracting forces.

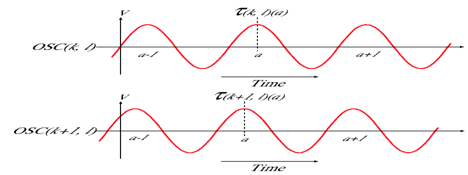

Fig. 4. The detection method of frequencies and the phase differences.

direction and for horizontal direction needs to be considered, because this system is 2 dimensional array. Therefore, three type synchronizations are observed as follows:

1) OSC(k, l)–OSC(k, $l+1$), OSC(k, l)–OSC(k, $l-1$), OSC(k, l)–OSC($k+1$, l), and OSC(k, l)–OSC($k-1$, l): in-phase synchronization.
2) {(OSC(k, l)–OSC(k, $l+1$) and OSC(k, l)–OSC(k, $l-1$): in-phase synchronization. OSC(k, l)–OSC($k+1$, l), and OSC(k, l)–OSC($k-1$, l): anti-phase synchronization.} **or** {OSC(k, l)–OSC(k, $l+1$), and OSC(k, l)–OSC(k, $l-1$): anti-phase synchronization. OSC(k, l)–OSC($k+1$, l), and OSC(k, l)–OSC($k+1$, l): int-phase synchronization.}
3) OSC(k, l)–OSC(k, $l+1$), OSC(k, l)–OSC(k, $l-1$), OSC(k, l)–OSC($k+1$, l), and OSC(k, l)–OSC($k+1$, l): anti-phase synchronization.

In this paper, we call the 1st type synchronization "in-and-in-phase synchronization." The 2nd type synchronization is called "in-and-anti-phase synchronization." The 3rd type synchronization is called "anti-and-anti-phase synchronization." An each instantaneous frequency $f_{(k,l)}$ of OSC(k, l) is obtained in each synchronization type. In the 1st situational synchronization, $f_{(k,l)}$ is f_{in-in}. In the 2nd situational synchronization, $f_{(k,l)}$ is $f_{in-anti}$. In the 3rd situational synchronization, $f_{(k,l)}$ is $f_{anti-anti}$.

These behaviors are shown in Table I. There is a disappearance by collision of two phase-inversion waves other than these phenomena.

V. Mechanism

We analyze about the penetration and reflection of the two phase-inversion waves.

The phase-inversion wave shows in Fig.2. The mechanisms are made clear using instantaneous frequency of each oscillator and phase difference between adjacent oscillators. The coupling parameter is fixed as $\alpha = 0.01$, and nonlinearity is fixed as $\varepsilon = 0.05$. An equation of the instantaneous frequency

TABLE I
CHARACTERISTICS OF THE PHASE-INVERSION WAVES.

Names of behaviors	*Itinerancies of instantaneous frequencies*	*Phenomena*
Propagations	$f_{in-in} \Leftrightarrow f_{in-anti}$, & $f_{in-anti} \Leftrightarrow f_{anti-anti}$	The phase-inversion waves propagate to vertical direction or horizontal direction. The vertical phase-inversion waves move from the horizontal phase-inversion waves independently.
Penetrations	$f_{in-in} \Leftrightarrow f_{anti-anti}$	Two phase-inversion waves arrives at an oscillator from vertical direction and horizontal direction, and each phase-inversion wave penetrates each other.
Reflections at an edge	$f_{in-in} \Leftrightarrow f_{in-anti}$, & $f_{in-anti} \Leftrightarrow f_{anti-anti}$	When an phase-inversion wave arrives at an edge, the phase-inversion wave reflects and propagates to where they came from. Sometime this phenomenon is happened with penetration.
Reflections between two phase-inversion waves	$f_{in-in} \Leftrightarrow f_{in-anti}$, & $f_{in-anti} \Leftrightarrow f_{anti-anti}$	When two phase-inversion waves coming from the opposite directions arrive to two adjacent oscillator at same time, the phase-inversion waves reflect and propagate to where they came from.

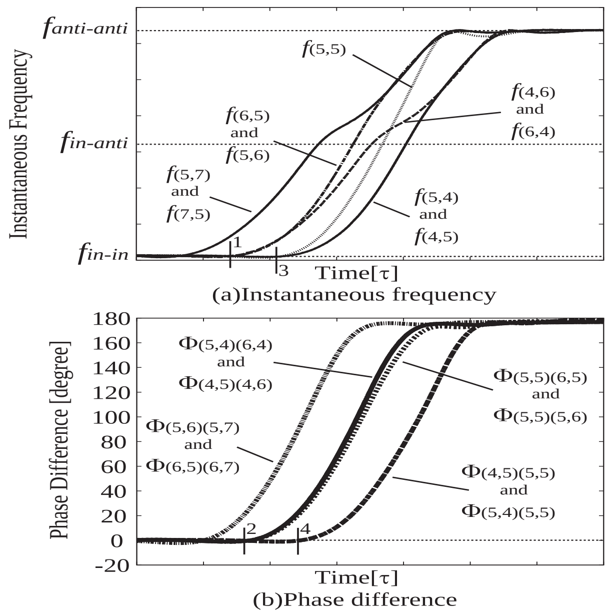

Fig. 5. Transitions of phase difference and frequencies by penetration a phase-inversion wave.

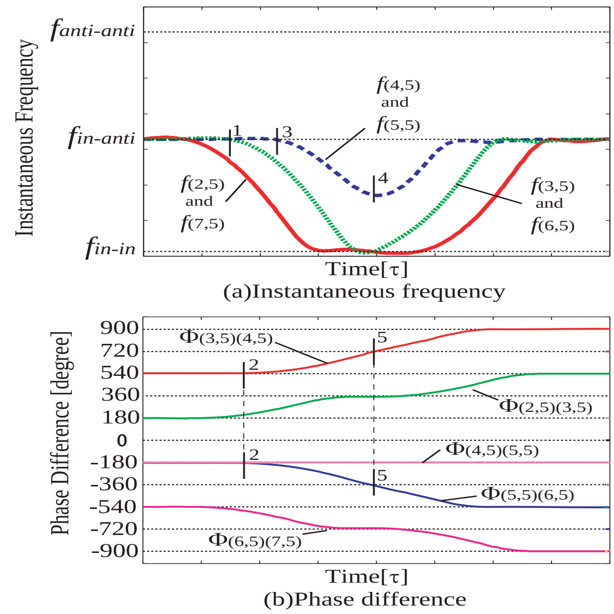

Fig. 6. Transitions of phase difference and frequencies by reflection between two phase-inversion wave.

of OSC(k, l) is calculated as follows. The instantaneous frequency is named $f_{(k,l)}(a)$ where "a" expresses the number of times of the peak value of the voltage. Time of a peak value of the voltage of OSC(k, l) is assumed as $\tau_{(k,l)}(a)$(see Fig.4). Similarly, $\tau_{(k+1,l)}(a)$ and $\tau_{(k,l+1)}(a)$ are decided. The $f_{(k,l)}(a)$ is obtained by Eq.(7).

$$f_{(k,l)}(a) = \frac{1}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)}. \tag{7}$$

The phase difference is calculated as follows. A phase difference between OSC(k, l) and OSC($k+1$, l) and a phase difference between OSC(k, l) and OSC(k, $l+1$) are calculated. The phase differences are assumed as $\Phi_{(k,l)(k+1,l)}(a)$ and $\Phi_{(k,l)(k,l+1)}(a)$ respectively(see Fig.4). The $\Phi_{(k,l)(k+1,l)}(a)$ and $\Phi_{(k,l)(k,l+1)}(a)$ are obtained by Eq.(8).

$$\begin{aligned} \Phi_{(k,l)(k+1,l)}(a) &= \frac{\tau_{(k,l)}(a) - \tau_{(k+1,l)}(a)}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)} \times 360 \text{ [degree]} \\ \Phi_{(k,l)(k,l+1)}(a) &= \frac{\tau_{(k,l)}(a) - \tau_{(k,l+1)}(a)}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)} \times 360 \text{ [degree]}. \end{aligned} \tag{8}$$

A. Penetration Mechanism

We can observe the phenomenon of penetration. A mechanism of penetration is shown in Tab. II and Fig. 5. The vertical axis of Fig. 5(a) is the instantaneous frequency, and horizontal direction is time. The vertical axis of Fig. 5(b) expresses the phase differences, and horizontal direction expresses time.

B. Reflection mechanisms between two phase-inversion waves

We can observe the reflection phenomenon between two phase-inversion waves which are arrives at two adjacent oscillators at same time from opposite directions respectively. We shows mechanism in Table III. In Fig.6(a), the vertical axis is the instantaneous frequencies, and the horizontal axis is time In Fig.6(b), the vertical axis is the phase differences, and the horizontal axis is the time.

VI. CONCLUSION

An attracting forces to in-phase and to anti-phase were made clear by all adjacent oscillators being fixed as arbitrary phase

TABLE II
PENETRATION MECHANISM(SEE FIG.5).

no.	*Vertical direction*	*Horizontal direction*
0	Now, phase-inversion waves, which changes phase states from in-phase synchronization to anti-phase synchronization, are arrived at oscillators of 7th row line from oscillators of 8th row line.	Now, phase-inversion waves, which changes phase states from in-phase synchronization to anti-phase synchronization, are arrived at oscillators of 7th column line from oscillators of 8th column line.
1	$\Phi_{(6,5)(7,5)}$ is increasing toward 180 degrees. $f_{(6,5)}$ slowly starts to increase from f_{in-in} to $f_{in-anti}$, after $\Phi_{(6,5)(7,5)}$ start to increase, because an attracting force to in-phase is weak around 0 degree(see Fig.3). $f_{(6,4)}$ also starts to increase from f_{in-in}, because a phase-inversion wave of adjacent vertical direction arrives at 6th row line at the same time. $f_{(6,4)}$ changes toward $f_{in-anti}$, because horizontal wave does not arrives at OSC(6,4) yet.	$\Phi_{(5,6)(5,7)}$ is increasing toward 180 degrees. f(5,6) slowly starts to increase from f_{in-in} to $f_{in-anti}$, after $\Phi_{(5,6)(5,7)}$ start to increase, because an attracting force to in-phase is weak around 0 degree(see Fig.3). $f_{(4,6)}$ also starts to increase from f_{in-in}, because a phase-inversion wave of adjacent vertical direction arrives at 6th column line at the same time. $f_{(4,6)}$ changes toward $f_{in-anti}$, because horizontal wave does not arrives at OSC(4,6) yet.
2	$\Phi_{(5,5)(6,5)}$ and $\Phi_{(5,4)(6,4)}$ start to increase toward 180 degrees, because $f_{(6,5)}$ and $f_{(6,4)}$ increase toward $f_{in-anti}$.	$\Phi_{(5,5)(5,6)}$ and $\Phi_{(4,5)(4,6)}$ start to increase toward 180 degrees, because $f_{(5,6)}$ and $f_{(4,6)}$ increase toward $f_{in-anti}$.
3	A horizontal phase-inversion wave and a vertical phase-inversion wave arrives at OSC(5,5). $f_{(5,5)}$ starts to increase to $f_{anti-anti}$ by horizontal phase-inversion wave and vertical phase-inversion wave rapidly. $f_{(4,5)}$ and $f_{(5,4)}$ start to increase from f_{in-in} to $f_{anti-anti}$, because $\Phi_{(5,4)(6,4)}$ and $\Phi_{(4,5)(4,6)}$ start to increase toward 180 degrees.	
4	Therefore, $\Phi_{(4,5)(5,5)}$ starts to increase toward 180 degrees because $f_{(5,5)}$ increases toward $f_{anti-anti}$. Because an attracting force to anti-phase is weak around 180 degrees, $\Phi_{(4,5)(5,5)}$ arrives at 180 degrees slowly(see Fig.3).	Therefore, $\Phi_{(5,4)(5,5)}$ starts to increase toward 180 degrees because $f_{(5,5)}$ increases toward $f_{anti-anti}$. Because an attracting force to anti-phase is weak around 180 degrees, $\Phi_{(5,4)(5,5)}$ arrives at 180 degrees slowly(see Fig.3).
The phase-inversion waves penetrate each other by above mechanism.		

TABLE III
REFLECTION MECHANISM BETWEEN TWO PHASE-INVERSION WAVES(SEE FIG.6).

no.	*upside*	*down side*
0	Now the phase states of horizontal direction are fixed in in-phase synchronization. The phase states of vertical directions around OSC(5,5) are anti-phase synchronization The phase-inversion wave changing from anti-phase to in-phase synchronization arrives on 3rd row and 6th row line from 0th row line and 9th row line respectively.	
1	$f_{(3,5)}$ starts to decrease from $f_{in-anti}$ to f_{in-in}, because $\Phi_{(2,5)(3,5)}$ started to change from anti-phase to in-phase. $f_{(3,5)}$ slowly changes, because an attracting force is weak force around anti-phase synchronization(see Fig.3).	$f_{(6,5)}$ starts to decrease from $f_{in-anti}$ to f_{in-in}, because $\Phi_{(6,5)(7,5)}$ started to change from anti-phase to in-phase. $f_{(6,5)}$ slowly changes, because an attracting force is weak force around anti-phase synchronization(see Fig.3).
2	$\Phi_{(3,5)(4,5)}$ starts to changing from anti-phase to in-phase, because $f_{(3,5)}$ started to decrease from $f_{in-anti}$ to f_{in-in}.	$\Phi_{(5,5)(6,5)}$ starts to change from anti-phase to in-phase, because $f_{(5,5)}$ started to decrease from $f_{in-anti}$ to f_{in-in}.
3	$f_{(4,5)}$ starts to decrease from $f_{in-anti}$ to f_{in-in} slowly, because $\Phi_{(3,5)(4,5)}$ started to change from anti-phase to in-phase.	$f_{(5,5)}$ starts to decrease from $f_{in-anti}$ to f_{in-in} slowly, because $\Phi_{(5,5)(6,5)}$ started to change from anti-phase to in-phase.
4	$\Phi_{(4,5)(5,5)}$ are fixed anti-phase, because the frequencies of OSC(4,5) and OSC(5,5) are same itinerancy. Therefore, $f_{(4,5)}$ and $f_{(5,5)}$ stop between $f_{in-anti}$ and f_{in-in}, and $f_{(4,5)}$ and $f_{(5,5)}$ start to increase to $f_{in-anti}$ again.	
5	$\Phi_{(3,5)(4,5)}$ can not stop at in-phase synchronization, because reflection phase-inversion wave arrives in a moment again.	$\Phi_{(4,5)(5,5)}$ can not stop at in-phase synchronization, because reflection phase-inversion wave arrives in a moment again.
Two phase-inversion waves reflect by above mechanism.		

difference.

A penetration mechanism between the vertical phase-inversion wave and horizontal phase-inversion wave was made clear by instantaneous frequencies of each oscillator, phase differences between adjacent oscillators and the attracting forces. Furthermore, a reflection mechanism between two phase-inversion waves was made clear by instantaneous frequencies of each oscillator, phase differences between adjacent oscillators and the attracting forces.

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