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Synchronization in Three Coupled van der Pol Oscillators with Different Coupling Strength

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1. Introduction

High-dimensional nonlinear phenomena in the field of natural sciences can often be explained as the result of a synchronization phenomenon of coupled oscillatory systems. Therefore, studies of synchronization in oscillatory networks have been extensively reported in the various fields such as physical [1], biological [2] and electrical [3] systems. Endo et al. have presented the details of a theoretical analysis and corresponding circuit experiments on electrical circuits oscillators arranged in a ladder, a ring and in a two-dimensional array topology [4]-[6]. Moreover, coupled oscillatory systems can also produce interesting phase patterns, including wave propagation, clustering and complex patterns [7]-[10]. Setou et al. have observed interesting synchronization phenomena (oscillation death, independent oscillation and double mode oscillation) when van der Pol oscillators with different frequencies are coupled by means of a resistor, in a star topology [11]. We assume that any fluctuations such as different frequencies have one possibility to induce interesting synchronization phenomena in oscillatory networks.

In this study, we discuss the synchronization phenomena in three coupled van der Pol oscillators as a ring topology with different coupling strength. As a first step, we consider the simplest case that only one coupling strength is varied. Namely, the coupling strength between the first and the second oscillators has the different coupling strength to the others. By carrying out computer simulations, we confirm that the phase differences of the three oscillators depend on the coupling strength between the first and the second oscillators. These phenomena are also analyzed theoretically by using the averaging method. Furthermore, we investigate the power consumption in the whole system.

2. Circuit Model

The circuit model of three coupled van der Pol oscillator as ring topology is shown in Fig. 1. In this figure, β denotes the amplitude for changing the coupling strength between first and second oscillators.

In the computer simulations, we assume that the $v_k - i_{Rk}$ characteristics of the nonlinear resistor in each oscillator is

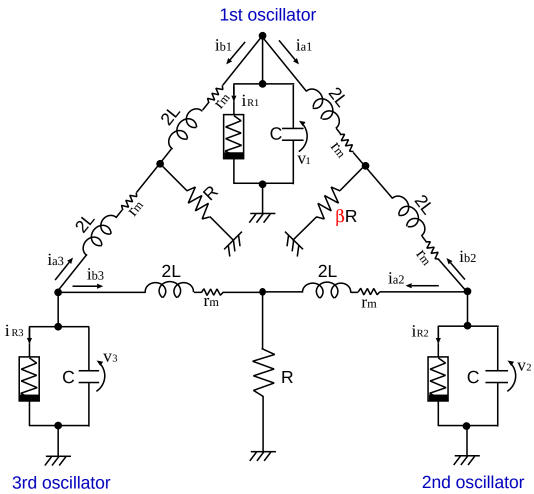

Figure 1: Three coupled van der Pol oscillators as a ring topology.

given by the following third order polynomial equation

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3 \quad (g_1, g_3 > 0), \quad (k = 1, 2, 3). \tag{1}$$

The normalized circuit equations governing the circuit in Fig. 1 are expressed as

[First oscillator]

$$\begin{cases} \dfrac{dx_1}{d\tau} = \varepsilon\left(1 - \dfrac{1}{3}{x_1}^2\right)x_1 - (y_{a1} + y_{b1}) \\ \dfrac{dy_{a1}}{d\tau} = \dfrac{1}{2}\left\{x_1 - \eta y_{a1} - \beta\gamma(y_{a1} + y_{b2})\right\} \\ \dfrac{dy_{b1}}{d\tau} = \dfrac{1}{2}\left\{x_1 - \eta y_{b1} - \gamma(y_{a3} + y_{b1})\right\} \end{cases} \tag{2}$$

[Second oscillator]

$$\begin{cases} \dfrac{dx_2}{d\tau} = \varepsilon\left(1 - \dfrac{1}{3}{x_2}^2\right)x_2 - (y_{a2} + y_{b2}) \\ \dfrac{dy_{a2}}{d\tau} = \dfrac{1}{2}\left\{x_2 - \eta y_{a2} - \gamma(y_{a2} + y_{b3})\right\} \\ \dfrac{dy_{b2}}{d\tau} = \dfrac{1}{2}\left\{x_2 - \eta y_{b2} - \beta\gamma(y_{a1} + y_{b2})\right\} \end{cases} \tag{3}$$

[Third oscillator]

$$\begin{cases} \dfrac{dx_3}{d\tau} = \varepsilon\left(1 - \dfrac{1}{3}{x_3}^2\right)x_3 - (y_{a3} + y_{b3}) \\ \dfrac{dy_{a3}}{d\tau} = \dfrac{1}{2}\left\{x_3 - \eta y_{a3} - \gamma(y_{a3} + y_{b1})\right\} \\ \dfrac{dy_{b3}}{d\tau} = \dfrac{1}{2}\left\{x_3 - \eta y_{b3} - \gamma(y_{a2} + y_{b3})\right\} \end{cases} \quad (4)$$

where

$$t = \sqrt{LC}\tau, \quad v_k = \sqrt{\frac{g_1}{3g_3}}x_k,$$

$$i_{ak} = \sqrt{\frac{g_1}{3g_3}}\sqrt{\frac{C}{L}}y_{ak}, \quad i_{bk} = \sqrt{\frac{g_1}{3g_3}}\sqrt{\frac{C}{L}}y_{bk},$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}}, \quad \gamma = R\sqrt{\frac{C}{L}}, \quad \eta = r_m\sqrt{\frac{C}{L}},$$

$$(k = 1, 2, 3).$$

In this equations, γ is the coupling strength and ε denotes the nonlinearity of the oscillators.

3. Synchronization Phenomena

For the computer simulations, we calculate Eqs. (2)-(4) using a fourth-order Runge-Kutta method with the step size $h = 0.005$. The parameters of this circuit model are setting for $\varepsilon = 0.1$, $\gamma = 0.02$ and $\eta = 0.001$.

3.1. Phase Difference

First, in order to calculate the phase difference between oscillators, let us define the Poincaré section as $x_k < 0$ and $y_{ak} + y_{bk} = 0$ $(k = 1, 2, 3)$. Next, we introduce the following independent variables from the discrete data of $x_{kl}(n)$, $y_{akl}(n)$ and $y_{bkl}(n)$ on the Poincaé section. The phase difference between k-th and l-th oscillators is defined as following equation.

$$\varphi_{kl}(n) = \begin{cases} \pi - tan^{-1}\dfrac{y_{akl}(n) + y_{bkl}(n)}{x_{kl}(n)}, \\ \quad x_{kl}(n) \geq 0 \\ -tan^{-1}\dfrac{y_{akl}(n) + y_{bkl}(n)}{x_{kl}(n)}, \\ \quad x_{kl}(n) < 0 \quad and \quad y_{akl} + y_{bkl}(n) \geq 0 \\ 2\pi - tan^{-1}\dfrac{y_{akl}(n) + y_{bkl}(n)}{x_{kl}(n)}, \\ \quad x_{kl}(n) < 0 \quad and \quad y_{akl} + y_{bkl}(n) < 0 \\ \qquad (k, l = 1, 2, 3) \end{cases} \quad (5)$$

We carry out computer simulations passing one hundred thousand iterations of the Poincaré section. Figure 2 shows the computer simulation result of the phase difference. When β is smaller than 0.5, first and second oscillators are synchronized at in-phase and other oscillators are synchronized at anti-phase. By increasing β, phase difference between the first and the second oscillators close to anti-phase synchronization and other oscillators synchronize with 90° shift.

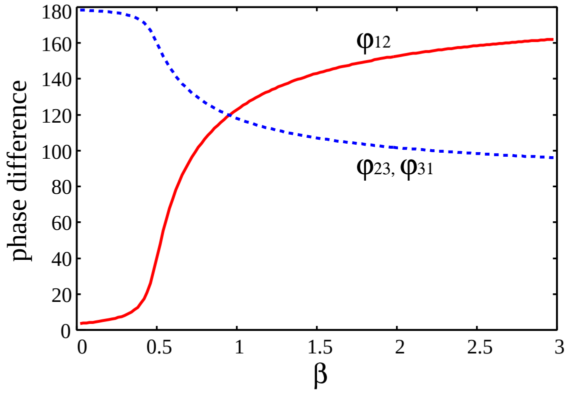

Figure 2: Phase differences.

3.2. Power Consumption of Resistors

In this section, we focus on the power consumption of the coupling resistors between the coupled oscillators. The average power P is calculated as follows,

$$\begin{aligned} P_{12} &= \frac{1}{T_p}\sum_{\tau=1}^{T_p}\beta\gamma(y_{a1} + y_{b2})^2 \\ P_{23} &= \frac{1}{T_p}\sum_{\tau=1}^{T_p}\gamma(y_{a2} + y_{b3})^2 \\ P_{31} &= \frac{1}{T_p}\sum_{\tau=1}^{T_p}\gamma(y_{a3} + y_{b1})^2 \\ P_{All} &= P_{12} + P_{23} + P_{31} \end{aligned} \quad (6)$$

where T_p is fixed for one thundered thousand as long period.

Figure 3 shows the power consumption of the coupling resistors. From this figure, we confirm that the energy of coupling strength between the first and the second oscillators P_{12} has the peak around $\beta = 0.5$. When β is smaller than 0.5, the power consumption of P_{23} and P_{31} shows zero. Because, in this case, these oscillators are synchronized at the anti-phase. By increasing the value of β, the energy of whole system (P_{All}) increases.

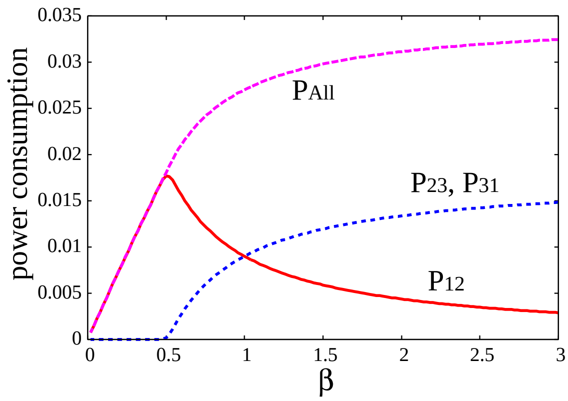

Figure 3: Power consumption.

4. Theoretical Analysis

4.1. Case for $\beta = 1.0$

Consider the case of that the three coupling strength has the same value ($\beta = 1.0$) and the tiny resistors to avoid L-loop for computer simulations is set to zero ($\eta = 0.0$). The normalized circuit equations are given as follows.

$$\begin{cases} \dfrac{dx_k}{d\tau} = \varepsilon\left(1 - \dfrac{1}{3}{x_k}^2\right)x_k - (y_{ak} + y_{bk}) \\ \dfrac{dy_{ak}}{d\tau} = \dfrac{1}{2}\left\{x_k - \gamma(y_{a(k-1)} + y_{bk})\right\} \\ \dfrac{dy_{bk}}{d\tau} = \dfrac{1}{2}\left\{x_k - \gamma(y_{ak} + y_{b(k+1)})\right\} \end{cases} \quad (k = 1, 2, 3) \tag{7}$$

where

$$y_{a0} = y_{a3}, \quad y_{b4} = y_{b1} \tag{8}$$

In this section, we analyze the synchronization phenomena by using the averaging method. Equation (6) is combined to give the second order nonliner differential equation as follows

$$\frac{d^2x_k}{d\tau^2} + x_k = \varepsilon(1 - {x_k}^2)\frac{dx_k}{d\tau} + \frac{1}{2}\gamma Y \equiv F_k \tag{9}$$

$$(k = 1, 2, 3)$$

$$\frac{dY}{d\tau} + \gamma Y = x_k + \frac{1}{2}x_{k-1} + \frac{1}{2}x_{k+1} \tag{10}$$

$$(k = 1, 2, 3)$$

where

$$Y \equiv y_{ak} + y_{bk} + y_{a(k-1)} + y_{b(k+1)}.$$

Equation (10) is first order linear differential equation. The solution is given as following equation.

$$Y = e^{-\gamma\tau}\int e^{\gamma\tau}\left(x_k + \frac{1}{2}x_{k-1} + \frac{1}{2}x_{k+1}\right) + Ce^{-\gamma\tau} \tag{11}$$

(C: const.)

In the steady state, the second term of Eq. (11) becomes to zero. Let us assume the solutions of Eq. (7) is

$$x_k(\tau) = \rho_k sin(\tau + \theta_k). \tag{12}$$

(k=1, 2, 3)

We pay attention to treat the non-resonance system and apply for the averaging method to Eq. (7). The average method is defined as following equations.

$$\dot{\rho}_k = \lim_{T\to\infty}\int_0^T \varepsilon F_k cos(\tau + \theta_k)d\tau$$
$$\dot{\theta}_k = \lim_{T\to\infty}\int_0^T \frac{\varepsilon}{\rho_k}F_k sin(\tau + \theta_k)d\tau \tag{13}$$

$$(k = 1, 2, 3)$$

By using Eq. (13), we obtain the solution of Eq. (7).

$$\begin{aligned} \dot{\rho}_k &= -\frac{\varepsilon^2\rho_k}{8}(\rho_k^2 - 4) \\ &-\frac{\varepsilon\gamma}{8(\gamma^2+1)}\Big\{2\rho_k + \gamma\rho_{k-1}sin(\theta_k - \theta_{k-1}) \\ &-\rho_{k-1}cos(\theta_k - \theta_{k-1}) + \gamma\rho_{k+1}sin(\theta_k - \theta_{k+1}) \\ &-\rho_{k+1}cos(\theta_k - \theta_{k+1})\Big\} \end{aligned} \tag{14}$$

$$\begin{aligned} \dot{\theta}_k &= -\frac{\varepsilon\gamma}{8\rho_k(\gamma^2+1)}\Big\{2\gamma\rho_k + \gamma\rho_{k-1}cos(\theta_k - \theta_{k-1}) \\ &-\rho_{k-1}sin(\theta_k - \theta_{k-1}) + \gamma\rho_{k+1}cos(\theta_k - \theta_{k+1}) \\ &-\rho_{k+1}sin(\theta_k - \theta_{k+1})\Big\} \end{aligned} \tag{15}$$

$$(k = 1, 2, 3).$$

In the steady state,

$$\dot{\rho}_k = 0 \quad and \quad \dot{\theta}_k = 0 \tag{16}$$

must be satisfied. We obtain the solutions as follows. For the amplitude:

$$\rho_k = \sqrt{4 - \frac{\gamma}{\varepsilon(\gamma^2+1)}} \tag{17}$$

$$(k = 1, 2, 3).$$

For the phase difference:

$$\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{2\pi}{3}. \tag{18}$$

In order to confirm the credibility of theoretical analysis, we compare the amplitude of oscillator and the phase difference of the theoretical result with the computer simulation result (see. Fig. 4 and Tab. 1). From these results, we confirm that they match very well.

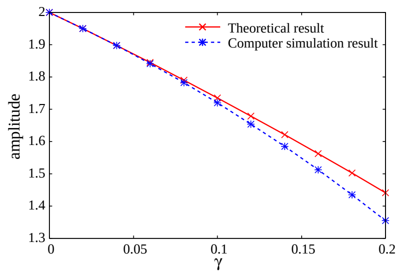

Figure 4: Comparison between theoretical and computer simulation results for the amplitude of oscillator.

Table 1: Comparison between theoretical and computer simulation results for the phase difference ($\theta_1 - \theta_2$).

	Theory	Simulation
Phase difference	120°	122°

4.2. Case for $\beta \neq 1.0$

Next we consider the case of that the coupling strength between the first and the second oscillators are changed ($\beta \neq 1.0$). The solutions of phase differences including the β factor read to

$$\begin{cases} -1 = \beta cos(\theta_1 - \theta_2) + cos(\theta_2 - \theta_3) \\ -1 = \beta cos(\theta_1 - \theta_2) + cos(\theta_3 - \theta_2) \\ -1 = cos(\theta_2 - \theta_3) + cos(\theta_3 - \theta_2) \end{cases} \quad (19)$$

Namely each difference phase shift is obtained.

$$\begin{aligned} \theta_1 - \theta_2 &= arccos\left(-\frac{1}{2\beta}\right) \\ \theta_2 - \theta_3 &= -1 - cos(\theta_1 - \theta_2) \\ \theta_3 - \theta_1 &= -1 - cos(\theta_1 - \theta_2) \end{aligned} \quad (20)$$

Finally, we compare the theoretical method and the computer simulations for the phase differences when the coupling strength between the first and the second oscillator (β) is changed from 0.02 to 3.0 (see. Figs. 5, 6). From Fig. 5 we can see that the phase difference between the theoretical and the computer simulation results has small difference around 10 degree. In the case of Fig. 6, they match perfectly.

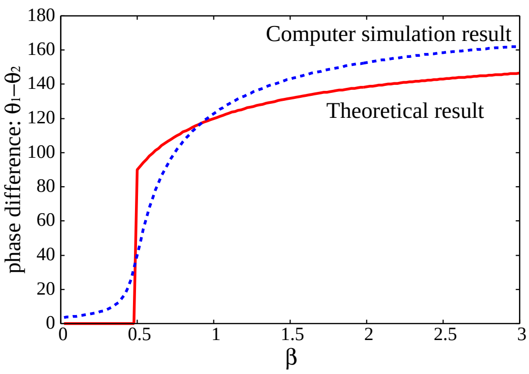

Figure 5: Phase difference ($\theta_1 - \theta_2$).

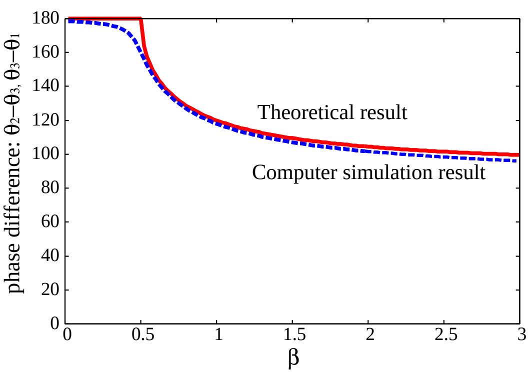

Figure 6: Phase difference ($\theta_2 - \theta_3$, $\theta_3 - \theta_1$).

5. Conclusions

We have investigated the synchronization phenomena in a ring van der Pol oscillators when the symmetric property of network is destroyed. By computer simulations, we have confirmed that three coupled oscillators tend to synchronize to be minimum energy of the whole system. Furthermore, we have applied the theoretical method to these synchronization phenomena and the similar solutions for the amplitude and the phase difference were obtained.

In our future works, we discuss the stability of solutions and explain the synchronization mechanism of a large-scale network by using this basic circuit model.

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