Mechanisms of Disappearance between Two Phase-Inversion Waves and Reflection of Two Phase-Inversion Waves at a Corner on 2D Lattice Oscillators

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Abstract
We observe and analyze synchronization phenomena on coupled oscillators system. We discovered the phase-inversion wave, which phenomena of changing phase states between two adjacent oscillators from in-phase synchronization to anti-phase synchronization or from anti-phase synchronization to in-phase synchronization. The phase-inversion wave behaviors observed propagation, penetration, reflection, and disappearance. This paper clarifies disappearance mechanism between two phase-inversion waves, and reflection mechanism when two phase-inversion waves arrive at a corner on a lattice of coupled oscillators system at same time.

1. Introduction
A lot of synchronization phenomena can be observed and studied in various fields[1]-[2]. For example, the laser is high-intensity ray bundle, the synchronization of fireflies, and so on. In our previous study, we observed and analyzed synchronization phenomena between adjacent oscillators on coupled oscillators system as a ladder and a lattice[3]-[5]. We discovered the phase-inversion wave, which phenomena of changing phase states between two adjacent oscillators from in-phase synchronization to anti-phase synchronization or from anti-phase synchronization to in-phase synchronization. We analyzed propagation mechanism of the phase-inversion wave, and reflection mechanism between two phase-inversion waves.

In this paper, we clarify disappearance mechanisms between two phase-inversion waves, and reflection mechanism when two phase-inversion waves arrive at a corner at same time using instantaneous frequency of each oscillators and phase differences between adjacent oscillators on the lattice system.

2. Circuit model
A lot of van der Pol oscillators are coupled by inductors $L_0$ as a lattice (see Fig. 1). The number of column and the number of row of this system are assumed as “$M + 1$” and “$N + 1$” respectively. We name each oscillator OSC($k,l$). A voltage of each oscillator is named $V_i(k,l)$, and a current of inductor of each oscillator is named $i_s(k,l)$ (see Fig. 1). The circuit equations of this circuit model are normalized by Eq. (1), and the normalized circuit equations are shown as Eqs. (2)-(6).

$\frac{dV_i(k,l)}{dt} = \frac{g_s}{L_0} i_s(k,l)$
$V_i(k,l) = \frac{g_t}{L_0} i_s(k,l)$
$t = \sqrt{L_0 C_0}$
$\frac{di_s(k,l)}{dt} = \alpha i_s(k,l) + \beta V_i(k,l) - \eta i_s(k,l) - 3 |i_s(k,l)|$ (1)

$\frac{dV_i(k,l)}{dt} = \frac{g_s}{L_0} i_s(k,l)$
$V_i(k,l) = \frac{g_t}{L_0} i_s(k,l)$
$t = \sqrt{L_0 C_0}$
$\frac{di_s(k,l)}{dt} = \alpha i_s(k,l) + \beta V_i(k,l) - \eta i_s(k,l) - 3 |i_s(k,l)|$ (2)

$\frac{dV_i(k-1,l)}{dt} = \frac{g_s}{L_0} i_s(k-1,l)$
$V_i(k-1,l) = \frac{g_t}{L_0} i_s(k-1,l)$
$t = \sqrt{L_0 C_0}$
$\frac{di_s(k-1,l)}{dt} = \alpha i_s(k-1,l) + \beta V_i(k-1,l) - \eta i_s(k-1,l) - 3 |i_s(k-1,l)|$ (3)

$\frac{dV_i(k,l-1)}{dt} = \frac{g_s}{L_0} i_s(k,l-1)$
$V_i(k,l-1) = \frac{g_t}{L_0} i_s(k,l-1)$
$t = \sqrt{L_0 C_0}$
$\frac{di_s(k,l-1)}{dt} = \alpha i_s(k,l-1) + \beta V_i(k,l-1) - \eta i_s(k,l-1) - 3 |i_s(k,l-1)|$ (4)

$\frac{dV_i(k+1,l)}{dt} = \frac{g_s}{L_0} i_s(k+1,l)$
$V_i(k+1,l) = \frac{g_t}{L_0} i_s(k+1,l)$
$t = \sqrt{L_0 C_0}$
$\frac{di_s(k+1,l)}{dt} = \alpha i_s(k+1,l) + \beta V_i(k+1,l) - \eta i_s(k+1,l) - 3 |i_s(k+1,l)|$ (5)

$\frac{dV_i(k,l+1)}{dt} = \frac{g_s}{L_0} i_s(k,l+1)$
$V_i(k,l+1) = \frac{g_t}{L_0} i_s(k,l+1)$
$t = \sqrt{L_0 C_0}$
$\frac{di_s(k,l+1)}{dt} = \alpha i_s(k,l+1) + \beta V_i(k,l+1) - \eta i_s(k,l+1) - 3 |i_s(k,l+1)|$ (6)

The $\alpha$ corresponds to the coupling parameter. The $\epsilon$ corresponds to the nonlinearity of each oscillator. This system is simulated by the fourth order Runge-Kutta method using Eqs. (2)-(6). The phase-inversion waves are shown in Fig. 2.

Figure 2-A expresses attractor of each oscillators. Figure 2-B expresses itinerary of phase difference sum of voltages of adjacent oscillators is shown along the time.
3. Attracting force

Attracting forces in in-phase or anti-phase synchronization are investigated on one parameter set where the phase-inversion waves can be observed (see Fig. 3). An equation of the instantaneous frequency of OSC(k, l) is obtained as follows. The instantaneous frequency is named \( f_{(i, j)}(a) \) where "a" expresses the number of times of the peak value of the voltage. Time of a peak value of the voltage of OSC(k, l) is assumed as \( \tau_{(i,j)}(a) \) (see Eq. (5)). Similarly, \( \tau_{(i,j)}(a) \) and \( \tau_{(i,j+1)}(a) \) are decided. The \( f_{(i,j)}(a) \) is obtained by Eq. (7).

\[
f_{(i,j)}(a) = \frac{1}{\tau_{(i,j)}(a) - \tau_{(i,j)}(a-1)}
\]  

The phase difference is calculated as follows. A phase difference between OSC(k, l) and OSC(k, l + 1) and a phase difference between OSC(k, l) and OSC(k, l + 1) are obtained. The phase differences are assumed as \( \Phi_{(0,0)}(a) \) and \( \Phi_{(0,0),(1,1)}(a) \) respectively (see Fig. 4). The \( \Phi_{(0,0)}(a) \) and \( \Phi_{(0,0),(1,1)}(a) \) are obtained by Eq. (8).

\[
\Phi_{(0,0)}(a) = \frac{\tau_{(i,j)}(a) - \tau_{(i+1,j)}(a)}{\tau_{(i+1,j)}(a) - \tau_{(i,j)}(a-1)} \times 360 \ [\text{degree}]
\]

\[
\Phi_{(0,0),(1,1)}(a) = \frac{\tau_{(i,j)}(a) - \tau_{(i+1,j+1)}(a)}{\tau_{(i+1,j+1)}(a) - \tau_{(i,j)}(a-1)} \times 360 \ [\text{degree}].
\]

Attracting forces are observed as follows:

1. The phase differences between all adjacent oscillators are fixed to arbitrary values as initial value in this lattice system.

2. The phase difference between OSC(1,4) and OSC(1,5) after one cycle from the initial value is analyzed changing the initial phase difference.

A vertical axis of Fig. 3 expresses a variation of phase difference per one cycle. An upper direction shows attracting force to anti-phase. A lower direction shows attracting force to in-phase. A horizontal axis shows initial phase differences. Therefore, the length of line shows a strength of attracting forces at each initial phase difference. Attracting force to in-phase is the strongest when the initial phase difference is -40 degrees. Attracting force to anti-phase is the strongest when the initial phase difference is -130 degrees.

4. Behavior of phase-inversion waves

We can observe some behaviors of phase-inversion waves on above systems. These behaviors are a propagation, a reflection at an edge, a reflection between phase-inversion waves, a penetration, a reflection at a corner, distinction and a disappearance. Moreover, these behaviors can be classified by frequencies, because three frequencies are observed in steady states. The synchronizations for vertical direction and for horizontal direction needs to be considered, because this system is 2 dimensional array. Therefore, three type synchronizations are observed as follows:

1. OSC(k, l)–OSC(k, l + 1), OSC(k, l)–OSC(k, l – 1), OSC(k, l)–OSC(k + 1, l), and OSC(k, l)–OSC(k – 1, l): in-phase synchronization.

2. OSC(k, l)–OSC(k, l + 1) and OSC(k, l)–OSC(k, l – 1): in-phase synchronization. OSC(k, l)–OSC(k + 1, l), and OSC(k, l)–OSC(k – 1, l): anti-phase synchronization. OSC(k, l)–OSC(k + 1, l), and OSC(k, l)–OSC(k – 1, l): in-phase synchronization. OSC(k, l)–OSC(k + 1, l), and OSC(k, l)–OSC(k – 1, l): anti-phase synchronization.

3. OSC(k, l)–OSC(k, l + 1), OSC(k, l)–OSC(k, l – 1), OSC(k, l)–OSC(k + 1, l), and OSC(k, l)–OSC(k – 1, l): anti-phase synchronization.

In this paper, we call the 1st type synchronization “in-and-in-phase synchronization.” The 2nd type synchronization is called “in-and-anti-phase synchronization.” The 3rd type synchronization is called “anti-and-anti-phase synchronization.” An each instantaneous frequency \( f_{(i,j)} \) of OSC(k, l) is obtained in each synchronization type. In the 1st situational synchronization, \( f_{(i,j)} \) is \( f_{\text{init}} \). In the 2nd situational synchronization, \( f_{(i,j)} \) is \( f_{\text{init}} \). In the 3rd situational synchronization,
Table 1: Characteristics of the phase-inversion waves.

<table>
<thead>
<tr>
<th>Names of behaviors</th>
<th>Itineraries of instantaneous frequencies</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propagations</td>
<td>$f_{in-in}$ $\Rightarrow$ $f_{out-out}$ &amp; $f_{in-out}$ $\Rightarrow$ $f_{out-in}$</td>
<td>The phase-inversion waves propagate to vertical direction or horizontal direction. The vertical phase-inversion waves move from the horizontal phase-inversion waves independently.</td>
</tr>
<tr>
<td>Penetrations</td>
<td>$f_{in-in}$ $\Rightarrow$ $f_{out-out}$</td>
<td>Two phase-inversion waves arrives at an oscillator from vertical direction and horizontal direction, and each phase-inversion wave penetrates each other.</td>
</tr>
<tr>
<td>Reflections at an edge</td>
<td>$f_{in-in}$ $\Rightarrow$ $f_{out-out}$ &amp; $f_{in-out}$ $\Rightarrow$ $f_{out-in}$</td>
<td>When a phase-inversion wave arrives at an edge, the phase-inversion wave reflects and propagates to where they come from. Sometimes this phenomenon is happened with penetration.</td>
</tr>
<tr>
<td>Reflections at a corner</td>
<td>$f_{in-in}$ $\Rightarrow$ middle of $f_{in-out}$ and $f_{out-in}$</td>
<td>When two phase-inversion waves coming from the vertical direction and the horizontal direction arrive at a corner oscillator at the same time, the phase-inversion waves reflect and propagate to where they come from.</td>
</tr>
<tr>
<td>Reflections between two phase-inversion waves</td>
<td>$f_{in-in}$ $\Rightarrow$ $f_{out-out}$ &amp; $f_{in-out}$ $\Rightarrow$ $f_{out-in}$</td>
<td>When two phase-inversion waves coming from the opposite directions arrive to adjacent oscillators at same time, the phase-inversion waves reflect and propagate to where they came from.</td>
</tr>
<tr>
<td>Disappearance</td>
<td>$f_{in-in}$ $\Rightarrow$ $f_{out-out}$</td>
<td>When two phase-inversion waves coming from the opposite directions arrive to one oscillator at same time, the phase-inversion waves disappear.</td>
</tr>
</tbody>
</table>

Figure 5: Disappearance of the phase-inversion waves on 11x11 oscillators.

Figure 6: Transitions of phase difference and frequencies by reflection at a corner of two phase-inversion waves.

Figure 7: Transitions of phase difference and frequencies by disappearance between two phase-inversion waves.

5. Mechanism

We analyze the reflection at a corner and the disappearance of the phase-inversion wave shows in Fig. 2 and Fig. 5. The mechanisms are made clear using instantaneous frequency of each oscillator and phase difference between adjacent oscillators. The coupling parameter is fixed as $\alpha = 0.01$, and non-linearity is fixed as $\varepsilon = 0.05$.

5.1. Reflection mechanism at a corner

We observe the reflection between two phase-inversion waves which are arrives at a corner oscillator at the same time from the vertical direction and the horizontal direction. A mechanism of reflection at a corner is shown in Tab. 2 and Fig. 6. In Fig. 6(a), the vertical axis is instantaneous frequencies, and horizontal axis is time. In Fig. 6(b), the vertical axis is the phase differences, and the horizontal axis is the time.

5.2. Disappearance mechanism

We can observe the disappearance. A mechanism of disappearance is shown in Tab. 3 and Fig. 7. In Fig. 7(a), the vertical axis is instantaneous frequencies, and horizontal axis is time. In Fig. 7(b), the vertical axis is the phase differences, and the horizontal axis is the time.

6. Conclusion

We observed and analyzed reflection mechanism when two phase-inversion waves arrive at a corner on a lattice...
of coupled oscillators systems. Furthermore, we observed and analyzed disappearance mechanism between two phase-inversion waves. These mechanisms are made clear using instantaneous frequency of each oscillator and phase difference between adjacent oscillators.

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References