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# Mechanisms of Disappearance between Two Phase-Inversion Waves and Reflection of Two Phase-Inversion Waves at a Corner on 2D Lattice Oscillators

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**Abstract**

We observe and analyze synchronization phenomena on coupled oscillators system. We discovered the phase-inversion wave, which phenomena of changing phase states between two adjacent oscillators from in-phase synchronization to anti-phase synchronization or from anti-phase synchronization to in-phase synchronization. The phase-inversion wave behaviors observed propagation, penetration, reflection, and disappearance. This paper clarifies disappearance mechanism between two phase-inversion waves, and reflection mechanism when two phase-inversion waves arrive at a corner on a lattice of coupled oscillators system at same time.

## 1. Introduction

A lot of synchronization phenomena can be observed and studied in various fields[1]-[2]. For example, the laser is high-intensity ray bundle, the synchronization of fireflies, and so on. In our previous study, we observed and analyzed synchronization phenomena between adjacent oscillators on coupled oscillators system as a ladder and a lattice[3]-[5]. We discovered the phase-inversion wave, which phenomena of changing phase states between two adjacent oscillators from in-phase synchronization to anti-phase synchronization or from anti-phase synchronization to in-phase synchronization. We analyzed propagation mechanism of the phase-inversion wave, and reflection mechanism between two phase-inversion waves.

In this paper, we clarify disappearance mechanism between two phase-inversion waves, and reflection mechanism when two phase-inversion waves arrive at a corner at same time using instantaneous frequency of each oscillators and phase differences between adjacent oscillators on the lattice system.

## 2. Circuit model

A lot of van der Pol oscillators are coupled by inductors $L_0$ as a lattice(see Fig. 1). The number of column and the number of row of this system are assumed as "$M+1$" and "$N+1$" respectively. We name each oscillator OSC($k$,$l$). A voltage of each oscillator is named $v_{(k,l)}$, and a current of inductor of each oscillator is named $i_{(k,l)}$(see Fig. 1). The circuit equations of this circuit model are normalized by Eq. (1), and the normalized circuit equations are shown as Eqs. (2)–(6).

$$i_{(k,l)} = \sqrt{\frac{Cg_1}{3Lg_3}}x_{(k,l)}, \quad v_{(k,l)} = \sqrt{\frac{g_1}{3g_3}}y_{(k,l)},$$
$$t = \sqrt{LC}\tau, \quad \frac{d}{d\tau} = \text{“ · ”}, \alpha = \frac{L}{L_0}, \quad \varepsilon = g_1\sqrt{\frac{L}{C}}. \tag{1}$$

[Corner–top] (left:$(a,b)$=(0, 1). right:$(a,b)$=$(N, N-1)$.)

$$\frac{dx_{(0,a)}}{d\tau} = y_{(0,a)}, \tag{2}$$
$$\frac{dy_{(0,a)}}{d\tau} = -x_{(0,a)} + \alpha(x_{(0,b)} + x_{(1,a)} - 2x_{(0,a)}) + \varepsilon(y_{(0,a)} - \frac{1}{3}y^3_{(0,a)}).$$

[Corner–bottom] (left:$(a,b)$=(0, 1). right:$(a,b)$=$(N, N-1)$.)

$$\frac{dx_{(M,a)}}{d\tau} = y_{(M,a)}, \tag{3}$$
$$\frac{dy_{(M,a)}}{d\tau} = -x_{(M,a)} + \alpha(x_{(M-1,a)} + x_{(M,b)} - 2x_{(M,a)}) + \varepsilon(y_{(M,a)} - \tfrac{1}{3}y^3_{(M,a)}).$$

[Center] ($0 < k < M$. $0 < l < N$.)

$$\frac{dx_{(k,l)}}{d\tau} = y_{(k,l)}, \tag{4}$$
$$\frac{dy_{(k,l)}}{d\tau} = -x_{(k,l)} + \alpha(x_{(k+1,l)} + x_{(k-1,l)} + x_{(k,l+1)} + x_{(k,l-1)} - 4x_{(k,l)}) + \varepsilon(y_{(k,l)} - \tfrac{1}{3}y^3_{(k,l)}).$$

[Edge]
(top:$(a,b)$=(0, 1).bottom:$(a,b)$=$(M, M-1)$.both:$0< l <N$.)

$$\frac{dx_{(a,l)}}{d\tau} = y_{(a,l)}, \tag{5}$$
$$\frac{dy_{(a,l)}}{d\tau} = -x_{(a,l)} + \alpha(x_{(a,l-1)} + x_{(a,l+1)} + x_{(b,l)} - 3x_{(a,l)}) + \varepsilon(y_{(a,l)} - \tfrac{1}{3}y^3_{(a,l)}).$$

(left:$(a,b)$=(0, 1). right:$(a,b)$=$(N, N-1)$. both:$0<k<M$.)

$$\frac{dx_{(k,a)}}{d\tau} = y_{(k,a)}, \tag{6}$$
$$\frac{dy_{(k,a)}}{d\tau} = -x_{(k,a)} + \alpha(x_{(k-1,a)} + x_{(k+1,a)} + x_{(k,b)} - 3x_{(k,a)}) + \varepsilon(y_{(k,a)} - \tfrac{1}{3}y^3_{(k,a)}).$$

The $\alpha$ corresponds to the coupling parameter. The $\varepsilon$ corresponds to the nonlinearity of each oscillator. This system is simulated by the fourth order Runge-Kutta method using Eqs. (2)-(6). The phase-inversion waves are shown in Fig. 2. Figure 2–A expresses attractor of each oscillators. Figure 2–B expresses itinerancy of phase difference sum of voltages of adjacent oscillators is shown along the time.

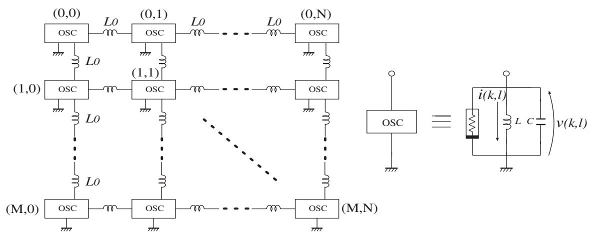


Figure 1: Circuit Model.

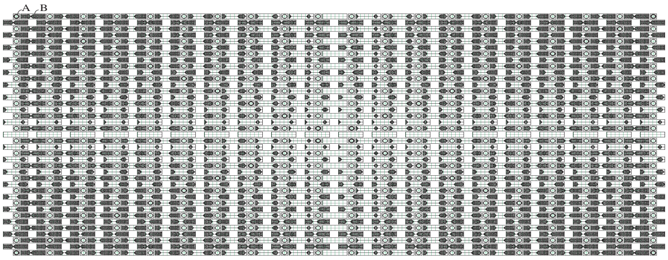

Figure 2: The phase-inversion waves on 20x20 oscillators(A:an attractor of each oscillator(current vs. voltage), B:a sum of voltages of adjacent oscillators(sum of voltage vs. time)).

## 3. Attracting force

Attracting forces to in-phase or anti-phase synchronization are investigated on one parameter-set where the phase-inversion waves can be observed(see Fig. 3). An equation of the instantaneous frequency of OSC($k$, $l$) is obtained as follows. The instantaneous frequency is named $f_{(k,l)}(a)$ where "a" expresses the number of times of the peak value of the voltage. Time of a peak value of the voltage of OSC($k$, $l$) is assumed as $\tau_{(k,l)}(a)$(see Fig. 4). Similarly, $\tau_{(k+1,l)}(a)$ and $\tau_{(k,l+1)}(a)$ are decided. The $f_{(k,l)}(a)$ is obtained by Eq. (7).

$$f_{(k,l)}(a) = \frac{1}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)}. \quad (7)$$

The phase difference is calculated as follows. A phase difference between OSC($k$, $l$) and OSC($k+1$, $l$) and a phase difference between OSC($k$, $l$) and OSC($k$, $l+1$) are obtained. The phase differences are assumed as $\Phi_{(k,l)(k+1,l)}(a)$ and $\Phi_{(k,l)(k,l+1)}(a)$ respectively(see Fig. 4). The $\Phi_{(k,l)(k+1,l)}(a)$ and $\Phi_{(k,l)(k,l+1)}(a)$ are obtained by Eq. (8).

$$\begin{aligned}\Phi_{(k,l)(k+1,l)}(a) &= \frac{\tau_{(k,l)}(a) - \tau_{(k+1,l)}(a)}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)} \times 360, \text{ [degree]} \\ \Phi_{(k,l)(k,l+1)}(a) &= \frac{\tau_{(k,l)}(a) - \tau_{(k,l+1)}(a)}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)} \times 360. \text{ [degree].}\end{aligned} \quad (8)$$

Attracting forces are observed as follows.

1. The phase differences between all adjacent oscillators are fixed to arbitrary values as initial value in this lattice system.
2. The phase difference between OSC(1,4) and OSC(1,5) after one cycle from the initial value is analyzed changing the initial phase difference.

A vertical axis of Fig. 3 expresses a variation of phase difference per one cycle. An upper direction shows attracting force to anti-phase. A lower direction shows attracting force to in-phase. A horizontal axis shows initial phase differences. Therefore, the length of line shows a strength of attracting

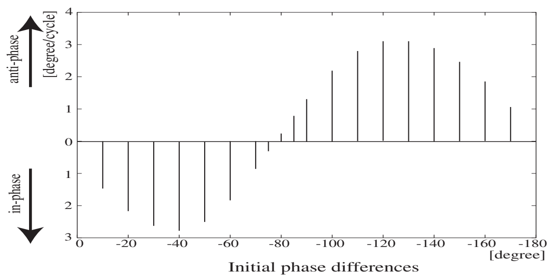


Figure 3: Attracting forces.

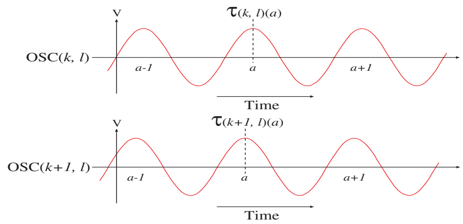


Figure 4: The detection method of frequencies and the phase differences.

forces at each initial phase difference. Attracting force to in-phase is the strongest when the initial phase difference is -40 degrees. Attracting force to anti-phase is the strongest when the initial phase difference is -130 degrees.

## 4. Behavior of phase-inversion waves

We can observe some behaviors of phase-inversion waves on above systems. These behaviors are a propagation, a reflection at an edge, a reflection between phase-inversion waves, a penetration, a reflection at a corner, distinction and a disappearance. Moreover, these behaviors can be classified by frequencies, because three frequencies are observed in steady states. The synchronizations for vertical direction and for horizontal direction needs to be considered, because this system is 2 dimensional array. Therefore, three type synchronizations are observed as follows:

1. OSC($k$, $l$)–OSC($k$, $l$+1), OSC($k$, $l$)–OSC($k$, $l$−1), OSC($k$, $l$)–OSC($k+1$, $l$), and OSC($k$, $l$)–OSC($k-1$, $l$): in-phase synchronization.
2. {(OSC($k$, $l$)–OSC($k$, $l+1$) and OSC($k$, $l$)–OSC($k$, $l-1$): in-phase synchronization. OSC($k$, $l$)–OSC($k+1$, $l$), and OSC($k$, $l$)–OSC($k-1$, $l$): anti-phase synchronization.} **or** {OSC($k$, $l$)–OSC($k$, $l+1$), and OSC($k$, $l$)–OSC($k$, $l-1$): anti-phase synchronization. OSC($k$, $l$)–OSC($k+1$, $l$), and OSC($k$, $l$)–OSC($k+1$, $l$): in-phase synchronization.}
3. OSC($k$, $l$)–OSC($k$, $l$+1), OSC($k$, $l$)–OSC($k$, $l$−1), OSC($k$, $l$)–OSC($k+1$, $l$), and OSC($k$, $l$)–OSC($k+1$, $l$): anti-phase synchronization.

In this paper, we call the 1st type synchronization "in-and-in-phase synchronization." The 2nd type synchronization is called "in-and-anti-phase synchronization." The 3rd type synchronization is called "anti-and-anti-phase synchronization." An each instantaneous frequency $f_{(k,l)}$ of OSC($k$, $l$) is obtained in each synchronization type. In the 1st situational synchronization, $f_{(k,l)}$ is $f_{in-in}$. In the 2nd situational synchronization, $f_{(k,l)}$ is $f_{in-anti}$. In the 3rd situational synchroniza-

Table 1: Characteristics of the phase-inversion waves.

| *Names of behaviors* | *Itinerancies of instantaneous frequencies* | *Phenomena* |
|---|---|---|
| Propagations | $f_{in-in} \Leftrightarrow f_{in-anti}$, & $f_{in-anti} \Leftrightarrow f_{anti-anti}$ | The phase-inversion waves propagate to vertical direction or horizontal direction. The vertical phase-inversion waves move from the horizontal phase-inversion waves independently. |
| Penetrations | $f_{in-in} \Leftrightarrow f_{anti-anti}$ | Two phase-inversion waves arrives at an oscillator from vertical direction and horizontal direction, and each phase-inversion wave penetrates each other. |
| Reflections at an edge | $f_{in-in} \Leftrightarrow f_{in-anti}$, & $f_{in-anti} \Leftrightarrow f_{anti-anti}$ | When a phase-inversion wave arrives at an edge, the phase-inversion wave reflects and propagates to where they came from. Sometime this phenomenon is happened with penetration. |
| Reflections at a corner | $f_{in-in}$ $\Leftrightarrow$ middle of $f_{in-anti}$ and $f_{anti-anti}$ | When two phase-inversion waves coming from the vertical direction and the horizontal direction arrive at a corner oscillator at the same time, the phase-inversion waves reflect and propagate to where they came from. |
| Reflections between two phase-inversion waves | $f_{in-in} \Leftrightarrow f_{in-anti}$, & $f_{in-anti} \Leftrightarrow f_{anti-anti}$ | When two phase-inversion waves coming from the opposite directions arrive to two adjacent oscillator at same time, the phase-inversion waves reflect and propagate to where they came from. |
| Disappearance | $f_{in-in} \Leftrightarrow f_{anti-anti}$ | When two phase-inversion waves coming from the opposite directions arrive to one oscillator at same time, the phase-inversion waves disappear. |

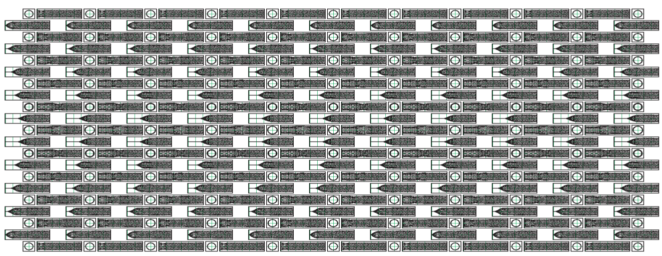

Figure 5: Disappearance of the phase-inversion waves on 11x11 oscillators.

tion, $f_{(k,l)}$ is $f_{anti-anti}$. These behaviors are shown in Table 1.

## 5. Mechanism

We analyze the reflection at a corner and the disappearance. The phase-inversion wave shows in Fig. 2 and Fig. 5. The mechanisms are made clear using instantaneous frequency of each oscillator and phase difference between adjacent oscillators. The coupling parameter is fixed as $\alpha = 0.01$, and nonlinearity is fixed as $\varepsilon = 0.05$.

### 5.1. Reflection mechanism at a corner

We observe the reflection between two phase-inversion waves which are arrives at a corner oscillator at the same time from the vertical direction and the horizontal direction. A mechanism of reflection at a corner is shown in Tab. 2 and Fig. 6. In Fig. 6(a), the vertical axis is instantaneous frequencies, and horizontal axis is time. In Fig. 6(b),the vertical axis is the phase differences, and the horizontal axis is the time.

### 5.2. Disappearance mechanism

We can observe the disappearance. A mechanism of disappearance is shown in Tab. 3 and Fig. 7. In Fig. 7(a), the vertical axis is instantaneous frequencies, and horizontal axis is time. In Fig. 7(b), the vertical axis is the phase differences, and the horizontal axis is the time.

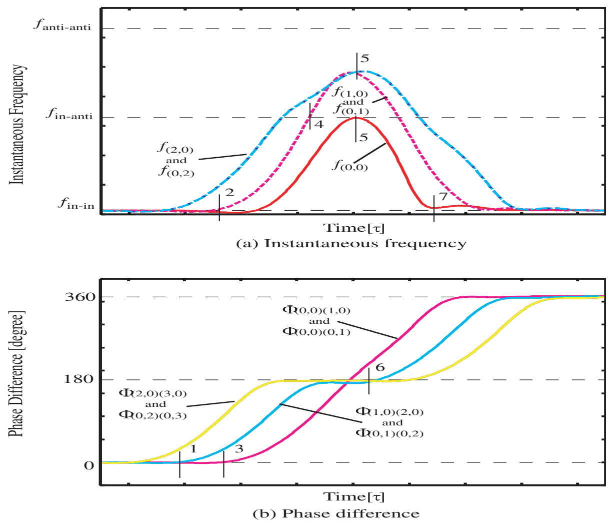


Figure 6: Transitions of phase difference and frequencies by reflection at a corner of two phase-inversion waves.

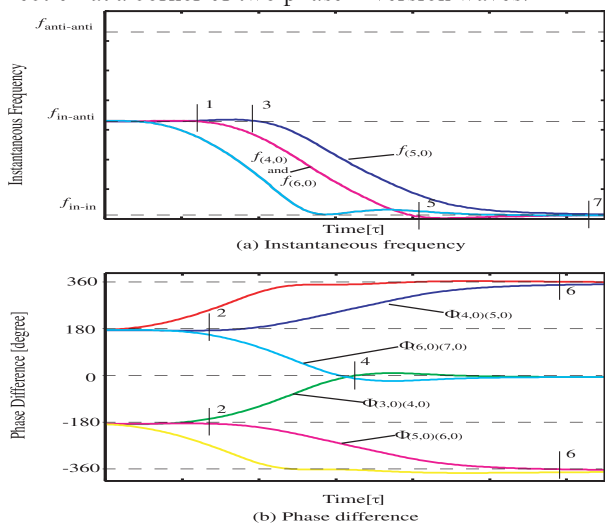


Figure 7: Transitions of phase difference and frequencies by disappearance between two phase-inversion wave.

## 6. Conclusion

We observed and analyzed reflection mechanism when two phase-inversion waves arrive at a corner on a lattice

Table 2: Reflection mechanism of two phase-inversion waves at a corner(see Fig. 6).

| *no.* | *vertical direction* | *horizontal direction* |
|---|---|---|
| 0 | Now the phase state between adjacent oscillators of the vertical direction is the in-phase synchronization. The phase-inversion waves, which change from in-phase synchronization to anti-phase synchronization, are arrived at OSC(2,0) from OSC(5,0). | Now the phase state between adjacent oscillators of the horizontal direction is the in-phase synchronization. The phase-inversion waves, which change from in-phase synchronization to anti-phase synchronization, are arrived at OSC(0,2) from OSC(0,5). |
| 1 | $\Phi_{(1,0)(2,0)}$ starts to increase toward 180 degrees from 0 degree, because $f_{(2,0)}$ starts to increase from $f_{in-in}$ toward $f_{in-anti}$. | $\Phi_{(0,1)(0,2)}$ starts to increase toward 180 degrees from 0 degree, because $f_{(0,2)}$ starts to increase from $f_{in-in}$ toward $f_{in-anti}$. |
| 2 | $f_{(1,0)}$ starts to increase from $f_{in-in}$ toward $f_{in-anti}$, because $\Phi_{(1,0)(2,0)}$ starts to increase toward 180 degrees from 0 degree. | $f_{(0,1)}$ starts to increase from $f_{in-in}$ toward $f_{in-anti}$, because $\Phi_{(0,1)(0,2)}$ starts to increase toward 180 degrees from 0 degree. |
| 3 | $\Phi_{(0,0)(1,0)}$ starts to increase toward 180 degrees from 0 degree, because $f_{(1,0)}$ starts to increase from $f_{in-in}$ toward $f_{in-anti}$. | $\Phi_{(0,0)(0,1)}$ starts to increase toward 180 degrees from 0 degree, because $f_{(0,1)}$ starts to increase from $f_{in-in}$ toward $f_{in-anti}$. |
| 4 | $f_{(1,0)}$ arrives around $f_{in-anti}$, because $\Phi_{(1,0)(2,0)}$ arrives around 180 degrees. $f_{(1,0)}$ starts to increase from $f_{in-anti}$ toward $f_{anti-anti}$. | $f_{(0,1)}$ arrives around $f_{in-anti}$, because $\Phi_{(0,1)(0,2)}$ arrives around 180 degrees. $f_{(0,1)}$ starts to increase from $f_{in-anti}$ toward $f_{anti-anti}$. |
| 5 | $f_{(0,0)}$ arrives around $f_{in-anti}$, because $\Phi_{(0,0)(1,0)}$ and $\Phi_{(0,0)(0,1)}$ arrive around 180 degrees at the same time. $\Phi_{(0,0)(1,0)}$ and $\Phi_{(0,0)(0,1)}$ start to increase toward 360 degrees from 180 degrees. The corner oscillator can not stable at anti-phase synchronization. Therefore, $f_{(0,0)}$ arrives around middle of $f_{in-in}$ and $f_{anti-anti}$. $f_{(1,0)}$, $f_{(2,0)}$, $f_{(0,1)}$, and $f_{(0,2)}$ arrive around middle of $f_{in-anti}$ and $f_{anti-anti}$. Therefore, $f_{(0,0)}$ starts to decrease toward $f_{in-in}$ again. $f_{(1,0)}$, $f_{(2,0)}$, $f_{(0,1)}$, and $f_{(0,2)}$ start to decrease toward $f_{in-anti}$ again. | |
| 6 | $\Phi_{(1,0)(2,0)}$ starts to increase toward 360 degrees from 180 degrees. | $\Phi_{(0,1)(0,2)}$ starts to increase toward 360 degrees from 180 degrees. |
| 7 | $f_{(0,0)}$ arrives around $f_{in-in}$. | |
| Two phase-inversion waves reflect at a corner by this mechanism. | | |

Table 3: Disappearance mechanism(see Fig. 7).

| *no.* | *upper side* | *lower side* |
|---|---|---|
| 0 | Now the phase state between adjacent oscillators of the horizontal direction is the in-phase synchronization. In vertical direction, the phase-inversion waves, which change from anti-phase synchronization to in-phase synchronization, are arrived at OSC(3,0) and OSC(7,0). | |
| 1 | $f_{(4,0)}$ starts to decrease from $f_{in-anti}$ toward $f_{in-in}$, because $\Phi_{(3,0)(4,0)}$ starts to increase toward 0 degree from -180 degrees. | $f_{(6,0)}$ starts to decrease from $f_{in-anti}$ toward $f_{in-in}$, because $\Phi_{(6,0)(7,0)}$ starts to decrease toward 0 degree from 180 degrees. |
| 2 | $\Phi_{(4,0)(5,0)}$ starts to increase toward 360 degrees from 180 degrees, because $f_{(4,0)}$ starts to decrease from $f_{in-anti}$ toward $f_{in-in}$. | $\Phi_{(5,0)(6,0)}$ starts to decrease toward -360 degrees from -180 degrees, because $f_{(6,0)}$ starts to decrease from $f_{in-anti}$ toward $f_{in-in}$. |
| 3 | $f_{(5,0)}$ starts to decrease from $f_{in-anti}$ toward $f_{in-in}$, because $\Phi_{(4,0)(5,0)}$ and $\Phi_{(5,0)(6,0)}$ start to change toward plus/minus 360 degrees from plus/minus 180 degrees. | |
| 4 | $\Phi_{(3,0)(4,0)}$ arrives around 0 degree, because $f_{(3,0)}$ arrives around $f_{in-in}$. The attracting force can not exist around 180 degrees. Therefore, 180 degrees is easy to stable. | $\Phi_{(6,0)(7,0)}$ arrives around 0 degree, because $f_{(7,0)}$ arrives around $f_{in-in}$. The attracting force can not exist around 180 degrees. Therefore, 180 degrees is easy to stable. |
| 5 | $f_{(4,0)}$ arrives around $f_{in-in}$, because $\Phi_{(3,0)(4,0)}$ arrives around 0 degree. | $f_{(6,0)}$ arrives around $f_{in-in}$, because $\Phi_{(6,0)(7,0)}$ arrives around 0 degree. |
| 6 | $\Phi_{(4,0)(5,0)}$ arrives around 360 degrees, because $f_{(4,0)}$ arrives around $f_{in-in}$. The attracting force can not exist around 0 degree. Therefore, 0 degree is easy to stable. | $\Phi_{(5,0)(6,0)}$ arrives around -360 degrees, because $f_{(6,0)}$ arrives around $f_{in-in}$. The attracting force can not exist around 0 degree. Therefore, 0 degree is easy to stable. |
| 7 | $f_{(5,0)}$ arrives around $f_{in-in}$, because $\Phi_{(4,0)(5,0)}$ and $\Phi_{(5,0)(6,0)}$ arrives around plus/minus 360 degrees. | |
| 8 | Each frequency fixes $f_{in-in}$, the phase differences fix in-phase synchronization. | |
| Two phase-inversion waves disappear by above mechanism. | | |

of coupled oscillators systems. Furthermore, we observed and analyzed disappearance mechanism between two phase-inversion waves. These mechanisms are made clear using instantaneous frequency of each oscillator and phase difference between adjacent oscillators.

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## References

[1] R. E. Mirollo and S. H. Strogatz "Synchronization of Pulse-Coupled Biological Oscillators," *SIAM Journal on Applied Mathematics*, vol. 50, no. 6, pp. 1645-1662, Dec. 1990.

[2] L. L. Bonilla, C. J. Pérez Vicente and R. Spigler, "Time-Periodic Phases in Populations of Nonlinearly Coupled Oscillators with Bimodal Frequency Distributions," *Physica D: Nonlinear Phenomena* vol. 113, issues 1, pp. 79–97, Feb. 1998.

[3] Y. Setou, Y. Inami, Y. Nishio and A. Ushida "On a Ladder of Oscillators Sharing Inductors," *Proc. of NOLTA'97*, vol. 1, pp. 565-568, Nov. 1997.

[4] M. Yamauchi, M. Okuda, Y. Nishio and A. Ushida, "Analysis of Phase-Inversion Waves in Coupled Oscillators Synchromizing at In-and-Anti-Phase," *IEICE Trans. Fundamentals*, vol.E86-A, no.7, pp.1799-1806, July. 2003.

[5] H. Aburatani, S. Yamane, M. Yamauchi and Y. Nishio "Reflection Mechanisms of a Phase-Inversion Wave at an Edge of 2D Lattice Oscillators," *Proc. NOLTA 2009*, pp.175-178, Oct. 2009.