Proceedings of the 40th ISCIE International Symposium
on Stochastic Systems Theory and Its Applications
Kyoto, Nov. 14-15, 2008.

Stochastic Analysis of Several Synchronization Phenomena on Coupled Chaotic Oscillators

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Abstract

Nonlinear oscillators including chaotic systems are very important devices, and furthermore it is one of essential component in the natural world to solve a mechanism of a nonlinear dynamics in several networks. In this study, a simple chaotic circuit with three states of both chaotic oscillation and two different size limit cycles which is called Multi-State Chaotic Circuit (MSCC) is proposed. Synchronization phenomena and complex behavior on a simple network system of the MSCCs coupled by some inductors are investigated. Several interesting chaotic phenomena of phase synchronization behavior have been observed in the coupled network system.

1 Introduction

An oscillator is an important device and one of essential component in the natural world. Nonlinear dynamics on coupled oscillators is considerable interesting for a wide variety of systems in several scientific fields and some engineering applications, especially in stochastic models. Although, many types of coupled circuit systems have been widely studied in order to clarify inherent features and many researchers have already proposed and investigated mechanism of them. The dynamics of multimode oscillations or phase synchronization on several coupled systems is still considerable interest from the viewpoint of both natural scientific fields and several applications. They have been confirmed in several systems; e.g., coupled van der Pol oscillators [1], laser systems [2], and so on. Phase synchronization and pattern dynamics are also interesting for several engineering applications.

On the other hand, many types of chaotic systems and circuits have already been proposed and investigated in detail. As interesting phenomena, there are famous chaotic attractors such a double-scroll family [3], n-double scroll [4]–[6] and scroll grid attractors [7]. If the active elements including in the systems have complexity constructed by compound some nonlinear elements, it can be easily considered that they yield several interesting features.

In our previous studies, the circuit which can individually behave both chaotic or periodic oscillations in the same parameters had been investigated [8]-[12]. This type of circuit was called a Multi-State Chaotic Circuit (abbr. MSCC). Multimode oscillations in coupled two or more multi-state chaotic circuits had been shown [9] on physical circuit experiments. Some complicated and interesting phenomena of phase synchronization had also been investigated [10]-[12]. It is known that complex behavior can be confirmed such chaotic itinerancy and spatio-temporal chaos on the large scale coupled networks. Some kinds of oscillation modes had been reported on large scale coupled chaotic circuits such phase synchronization, phase propagation and frustration of oscillation and so on [13]-[15]. On the other hand, coupled Van der Pol oscillators with hard nonlinearity had been investigated [16], and also stability analysis of them [17]. A study of coupled multi-state Van der Pol oscillator had been reported [18]. There are many oscillators from very low to very high frequency, which can be easily constructed on the real electrical circuits. We consider that it is important to investigate phase synchronization and pattern dynamics in such coupled oscillator systems.

This paper presents stochastic analysis for several phase synchronization phenomena of multi-state oscillators coupled by some inductors as several types of network. Each oscillator circuit can individually behave both periodic oscillations (limit cycles) and chaos in the same parameters. This proposed circuit can behave three states such as chaos and two different size of limit cycles when different initial conditions are given as initial states. In this study, we consider some coupled systems which each oscillator is connected to some other oscillators by inductors, and classification of phase synchronization modes is investigated. In numerical simulation, many types of phase synchronization modes are asynchronously confirmed in the proposed systems, however all parameters of each oscillator circuit are the same. It means that several phase synchronization modes are coexisting in the same parameters.

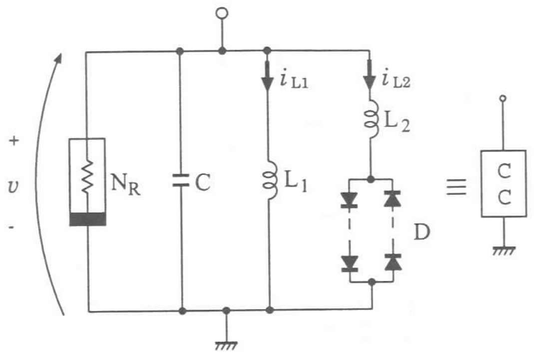

Fig. 1: Based chaotic circuit.

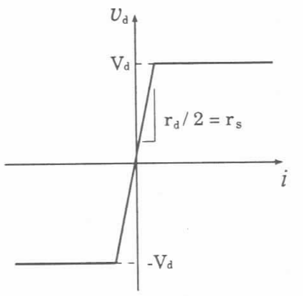

Fig. 2: Piecewise linear resistors.

2 Model Description

Figure 1 shows a basic chaotic circuit which consists three memory elements, diodes and nonlinear resistor. The basis of this circuit is proposed and analyzed by Inaba, et al [19]. The dynamics of the circuit has chaotic features. Although, this circuit diagram is a definitely system, observed phenomena are often stochastic behavior. The equations of the circuit as shown in Fig. 1 are described by

$$\begin{cases} L_1\dfrac{di_{L1}}{dt} &= v \\ L_2\dfrac{di_{L2}}{dt} &= v - v_d(i_{L2}) \\ C\dfrac{dv}{dt} &= -(i_{L1}+i_{L2}) - gv \end{cases} \tag{1}$$

where g is a linear negative conductance value of N_R, if we consider the negative resistor as an ideal active element. The $i-v$ characteristic of one diode is approximated by two segments piecewise linear functions. The part of diodes with polarities is constructed by some diodes, and their threshold voltages can be set as $+V_d$ and $-V_d$, respectively. Its characteristic is shown in Fig. 2 which is described as three segments piecewise linear functions by

$$v_d(i_{L2}) = \frac{1}{2}\{|r_s i_{L2} + V_d| - |r_s i_{L2} - V_d|\} \tag{2}$$

The variable $v_d(i_{L2})$ is a function depending on the current through their diodes D in Fig. 1, which determines their chaotic dynamics.

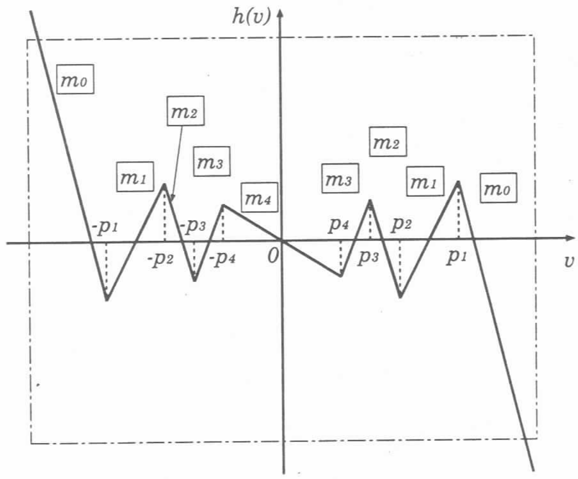

Fig. 3: Design for sawtooth nonlinear resistor N_R with respect to the origin.

By changing the following variables and parameters as follows:

$$\begin{aligned} & i_{L1} = \sqrt{\frac{C}{L_1}}V_d\,x\,,\quad i_{L2} = \sqrt{\frac{C}{L_1}}V_d\,y\,, \\ & v = V_d\,z\,,\quad t = \sqrt{L_1C}\,\tau\,,\quad \text{“}\cdot\text{”} = \frac{d}{d\tau}\,, \\ & \beta = \frac{L_1}{L_2}\,,\quad \gamma = g\sqrt{\frac{L_1}{C}}\,,\ \text{and}\ \delta = r_s\sqrt{\frac{C}{L_1}}\,. \end{aligned} \tag{3}$$

Then, we can obtain the normalized circuit equations with non-dimensional variables.

In this study, we substitute the negative resistor N_R including in the original chaotic circuit to a symmetrical continuous piecewise linear resistor. The definition of the sawtooth nonlinear resistor with both of slopes and break points are illustrated in Fig. 3. The proposed and designed chaotic circuit is the same as shown in Fig. 1. The piecewise linear resistor can be easily constructed by combining some components in parallel [8][9].

Let us consider that the part of negative resistance N_R is replaced to the function $h(z)$ represented by a voltage source z as a canonical form shown in Fig. 3. When we chose the threshold voltage V_d as a normalized value, then the circuit equations can be normalized and rewritten as follows:

$$\begin{cases} \dot{x} &= z \\ \dot{y} &= \beta(z - f(y)) \\ \dot{z} &= -(x+y) - \gamma^* h(z) \end{cases} \tag{4}$$

$$f(y) = \frac{1}{2}\{|\delta y + 1| - |\delta y - 1|\} \tag{5}$$

$$\begin{aligned} h(z) &= m_0 z + \frac{1}{2}\Big\{\sum_{k=0}^{K}(m_k - m_{k+1})\times \\ &\quad \{|z - p_{k+1}| - |z + p_{k+1}|\}\Big\} \end{aligned} \tag{6}$$

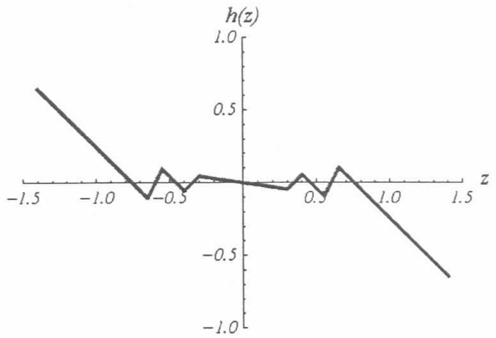

Fig. 4: Characteristic of a function $h(z)$ for the proposed circuit.

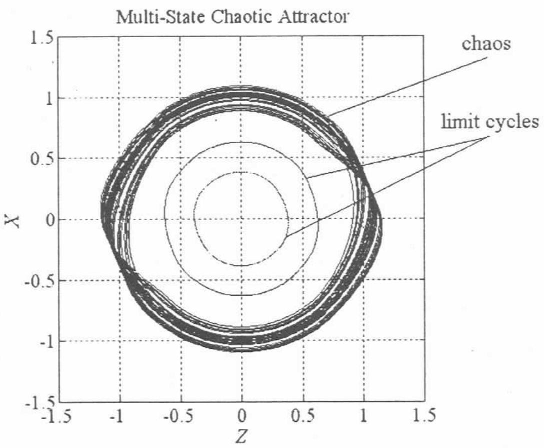

Fig. 5: Three states of attractors drawing onto the $z-x$ plane for the parameters $\beta = 10.0$, $\gamma^* = 0.78$ and $\delta = 100$ with the function $h(z)$ as a nonlinear resistor N_R.

where $f(y)$ and $h(z)$ are piecewise linear functions of the current y and the voltage z, respectively. The function $h(z)$ which is designed for several segments piecewise linear as symmetric with respect to the origin as shown in Fig. 3. The parameter γ^* is adopted as a common value for the magnitude of an entire shape, hence the values $m_k (k = 0,1,2,\cdots,K)$ mean magnitude of the slope to the ratio for γ^*. Hereafter, we set the parameters of the nonlinear function $h(z)$ fixed as follows:

$$\begin{aligned}&(p_1, p_2, p_3, p_4)\\ &\quad = (0.65, 0.55, 0.40, 0.30),\\ &(m_0, m_1, m_2, m_3, m_4)\\ &\quad = (-1.0, 2.0, -1.0, 1.0, -0.15).\end{aligned} \tag{7}$$

Then we can obtain the sawtooth shape shown in Fig. 4.

Figure 5 also shows a typical chaotic attractor obtained for the parameters $\beta = 10.0$, $\gamma^* = 0.78$, $\delta = 100$, with piecewise linear characteristics realized by (7). We can confirm that both chaotic and two periodic attractors coexist in the same parameters. It means that we can observe coexistence with both chaotic attractor and two different size of limit cycles in this circuit.

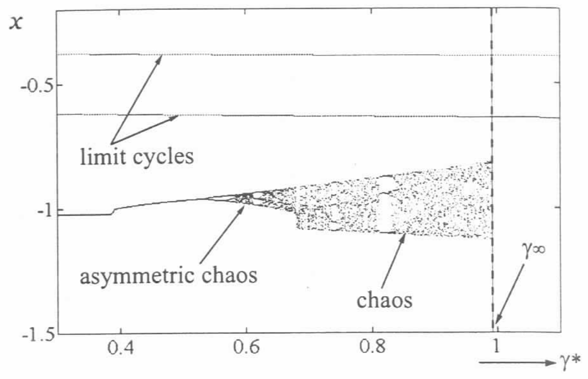

Fig. 6: Bifurcation diagram by changing the parameter γ^* for $\beta = 10.0$ and $\delta = 100$.

Further a bifurcation diagram by changing the parameter γ^* is shown in Fig. 6. As increasing γ^* periodic attractor bifurcates to chaos in the following routes while keeping the limit cycle at around the origin. Oscillation of symmetrical 1-period → asymmetrical 1-period → bifurcates to 2^n via period doubling bifurcation → asymmetrical slight chaos → symmetrical fluttered chaos. We can observe that both two oscillation modes exist separately in the same parameters. However, chaotic attractor disappear and limit cycle is only observed when γ^* is larger than around $0.98 (\equiv \gamma^\infty)$ in these parameters. The reason of this phenomenon is that dynamics of the circuit tends toward to be drawn into inside bounded area because trajectory is grown when the parameter γ^* becomes larger. Therefore, it will be stabilized to limit cycles.

3 Simulation of the coupled MSCCs

In this section, we consider some models of coupled network systems. There are many types of coupled systems, such a ring structure, a network, and so on. Figure 7 shows two examples of the coupled network systems.

First, let us consider the coupled MSCCs model which combined number of N chaotic circuits are connected by inductors L_0 as a ring structure as shown in Fig. 7(a). We use a new parameter $\alpha = L_1/L_0$ which corresponds to a coupling strength of both neighbors' circuit. Every chaotic circuit is composed by all the same parameters.

Therefore when we choose a threshold voltage value V_d as a criterion, the circuit equation of coupled MSCCs can be normalized by changing the variables (3) and the parameter α, then the entire circuit equations can be rewritten by

$$\left\{\begin{aligned} \dot{x}_k &= z_k\\ \dot{y}_k &= \beta(z_k - f(y_k))\\ \dot{z}_k &= \alpha(x_{k-1} - 2x_k + x_{k+1})\\ &\qquad -(x_k + y_k) - \gamma^* h(z_k)\end{aligned}\right. \tag{8}$$

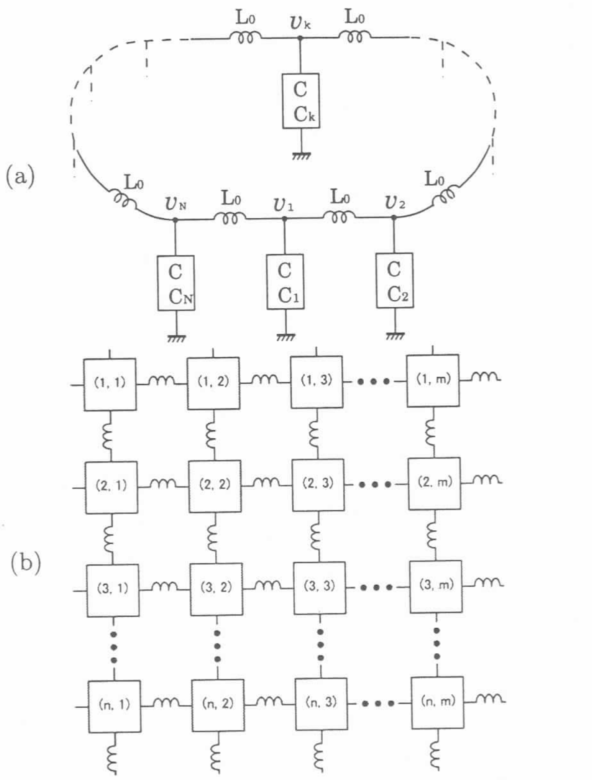

Fig. 7: Coupled network model: (a) ring type, (b) network type with neumann neighborhood.

Second, consider a model which an MSCC is connected four neighbors as a simple network as shown in Fig. 7(b). The size of networks is as (N, M), and the circuit on the edge of network is connected to opposite side circuit. It seems like a distribution on the surface of the torus structure. By the similar way as mentioned above, the circuit equations of the coupled MSCCs can be normalized and rewritten as follows:

$$\begin{cases} \dot{x}_{(i,j)} &= z_{(i,j)} \\ \dot{y}_{(i,j)} &= \beta(z_{(i,j)} - f(y_{(i,j)})) \\ \dot{z}_{(i,j)} &= \alpha\left(\sum_{\Phi} x_{\Phi} - 4x_{(i,j)}\right) \\ & \quad -(x_{(i,j)} + y_{(i,j)}) - h(z_{(i,j)}) \end{cases} \tag{9}$$

where Φ means a set of four neighbors to $x_{(i,j)}$.

We show some computer calculation results by using 4-th order Runge–Kutta method with time step size $\Delta t = 0.001$ for these circuit equations in some cases of the proposed network model. The parameters of each circuit are the same in the Sec. 2. The initial conditions of each circuit are supplied at random.

3.1 Two MSCCs case: $N = 2$

Now we consider that the number of the coupled MSCCs is two. This case is similar to the model in the sense of coexistence of chaotic and periodic oscillations for the previous work [9]. Although the detail results are omitted, we can confirm several types of phase synchronization modes in this model. In this case, some asynchronous oscillation modes could be confirmed consequently by numerical simulations when the initial conditions are varied. We could observe two different limit cycles by their oscillation size; in-phase synchronous limit cycles, anti-phase synchronous limit cycles, anti-phase chaotic synchronous state, and multimode oscillations in the same parameters.

3.2 Three or more MSCCs cases: $N \geq 3$

Let us consider the case of $N = 3$ or larger. The circuit parameters in each MSCC are set as all the same parameters described in the section 2 with an additional parameter $\alpha = 0.50$. Compare with the case $N = 2$, several different synchronization phenomena can be found. Because all types of the results can not be represented, some simulation results are only shown here in Fig. 8. From top of the figure, attractors drawing onto z–x plane, synchronization state of z_k–z_{k+1} plane, and waveform of difference between the two variables $z_k - z_{k+1}$. We could confirm to be coexisting a lot of phase synchronization modes.

In large coupled systems for $N \geq 4$, it is easily expected to be confirmed more complex behavior. In the case of $N = 4$, several types of synchronization modes are confirmed. Figure 8 (c) shows two-pair of in-phase and anti-phase synchronization of limit cycles. Further, figure (d) shows a new type of synchronization mode of phase locking phenomena. In the case of $N = 7$ or larger, no many synchronization modes have been confirmed in these parameters on this coupled system. Figure 8 (e) and (f) show typical simulation results for a large number of N. However when the number of N is large enough and furthermore the circuit parameters should be set appropriately, a certain kind of phase propagation phenomena may be confirmed. Thus, several complex behavior could be also confirmed as stochastic phenomena in the coupled multi-state chaotic oscillators.

Table 1 is a summary of the observation of several phase synchronization modes. When the number of MSCCs N is changing, it is indicated that several phase synchronization modes could be observed or not be observed. Each observation mode is stochastically independent by initial conditions which are supplied with random values. Several phase synchronization modes depend on the initial conditions. If the number of coupled MSCCs is very large, we can not easily expect to observe all synchronization modes.

3.3 Other cases of the network

We discuss a model of the network as shown in Fig. 7(b). The network size is as (N, M). If the case $(N, M) = (2, 2)$, this coupled system corresponds to a ring structure of the case $N = 4$ in the section 3.2. Figure 9 shows some typical results obtained from computer simulation in some cases of coupled number (N, M). The left side part of the figure shows attractors of each circuit, and the right side part indicates synchronization state between two oscillators. We can

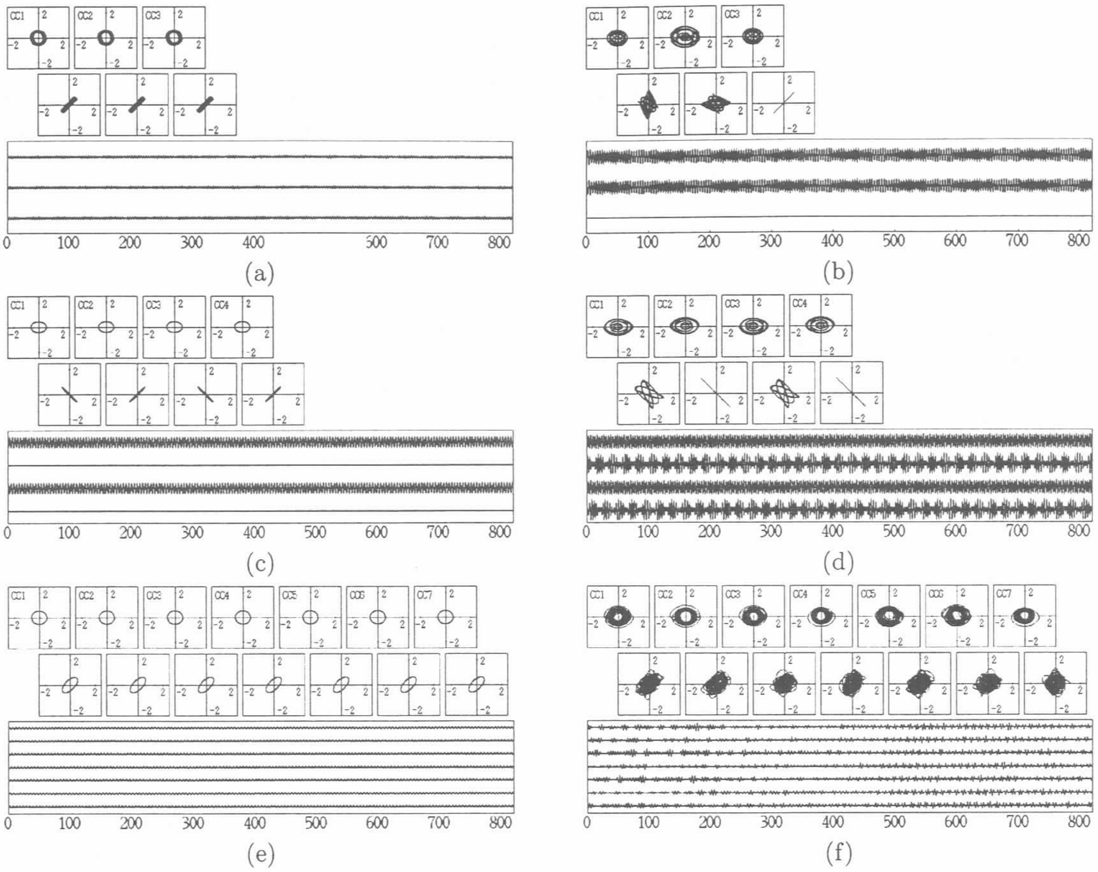

Fig. 8: Some simulation results obtained from coupled three, four and seven MSCCs for α = 0.50, β = 10.0, γ^* = 0.78, δ = 100. (a) in-phase synchronization, (b) double-mode and one-pair synchronization, (c) two-pair anti-phase synchronization, (d) two-pair double-mode and anti-phase synchronization of phase locking, (e) seven-phase synchronization, and (f) multimode oscillation.

confirm complex and chaotic synchronization phenomena on the coupled system.

Thus, we can confirm several types of complex and interesting synchronization phenomena in the same parameters on the coupled oscillators, i.e., in-phase synchronization, anti-phase synchronization, n-phase synchronization, phase locking, multimode oscillation, and other types. It is very interesting phenomena that several phase synchronization modes are coexisting in spite of the same parameters. Stochastic phenomena of these several synchronization modes which can be obtained from the definitely systems are investigated.

4 Conclusions

In this study, we have investigated several synchronization modes in coupled multi-state chaotic oscillator circuits. Coexistence of several types oscillation modes have been confirmed in coupled MSCCs. Moreover, stochastic analysis of several phase synchronization modes have been also shown. On large scale coupled chaotic oscillator such a small-world network and scale-free network, we consider that several types of complex behavior are expected to yield novel complicated phenomena e.g., spatio-temporal behavior or inherent emergent property.

Acknowledgment

This work was partly supported by GRANT-IN-AID for "Open Research Center" project from MEXT, Japan.

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Table 1: Summary of the observation of several phase synchronization modes on coupled MSCCs in the same parameters as shown in Fig. 8 and (7) when the number of MSCCs is changing from $N = 2$ to 7. Observable T: yes, F: no, E: uncertain. Observation types L: limit cycle (non-chaotic oscillation), C: chaos, Φ: in-phase synchronization, Θ: anti-phase synchronization, Λ_n: n-phase synchronization, Δ: multimode oscillations, O: other type, respectively.

	L-Φ	L-Θ	L-Λ_n	C-Φ	C-Θ	Δ	O
$N = 2$	T	T	-	F	T	T	-
$N = 3$	T	F	T	E	F	T	T
$N = 4$	T	T	T	E	F	T	T
$N = 5$	T	F	E	E	F	T	T
$N = 6$	T	T	T	E	F	E	E
$N = 7$	T	F	E	E	F	E	E

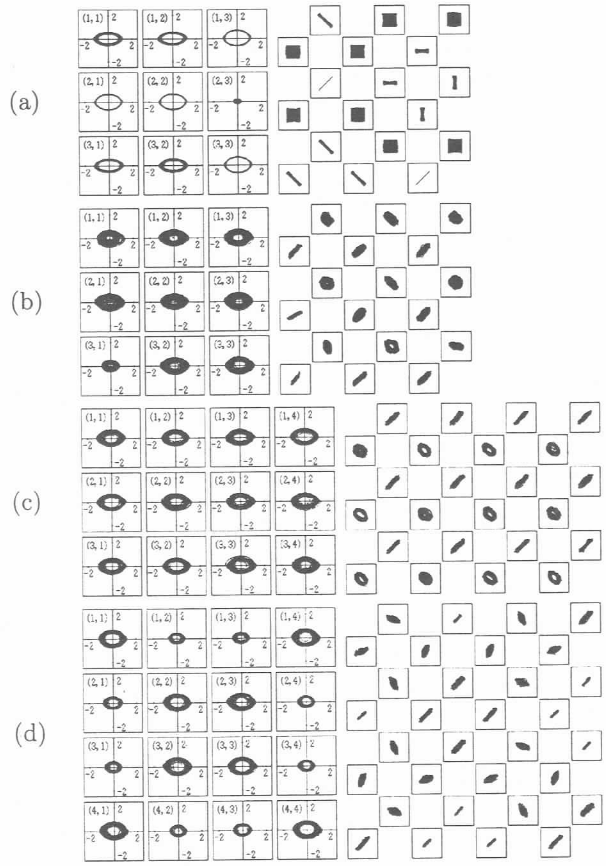

Fig. 9: Simulation results obtained from coupled MSCCs for $\alpha = 0.50$, $\beta = 10.0$, $\gamma^* = 0.78$, $\delta = 100$. Size (N, M) of the network: (a) (3,3), (b) (3,3), (c) (3,4) and (c) (4,4).

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