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Group Synchronization of van der Pol Oscillators with Different Frequencies

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Abstract—In this study, synchronization phenomena of van der Pol oscillators coupled by resistors in a ring are investigated, when the frequency error are added to some oscillators. Using computer simulations, we observe that the van der Pol oscillators with frequency error can produce the nonlinear phenomena related with synchronization such as oscillation death, independent oscillation and double-mode oscillation.

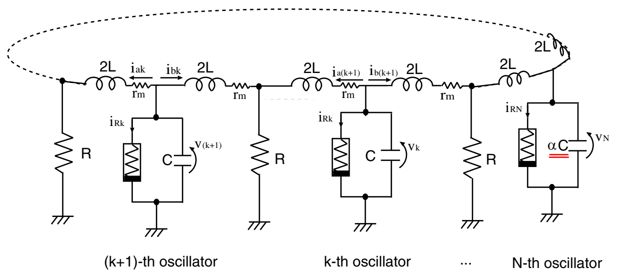
1. Introduction

Various synchronization phenomena observed in coupled oscillatory systems have the possibility to model high functional information processing of a human brain. Such oscillatory systems can produce the interesting phase pattern including: wave propagation, clustering, complex phase pattern [1]–[4]. Setou et al. have observed interesting synchronization phenomena when van der Pol oscillators with different oscillation frequencies are coupled by a resistor as a star structure [5]. However, it is difficult to investigate synchronization phenomena of large-scale circuits, because, van der Pol oscillators coupled in a star topology do not synchronize when the number of coupling oscillators is larger than 4.

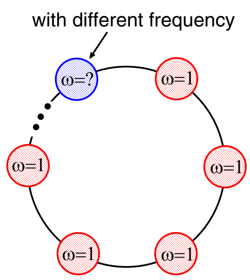
In this study, we consider a ring consisting of van der Pol oscillators with different oscillation frequencies for investigation of large-scale circuits. It is possible to add the frequency error to the van der Pol oscillators by setting the different value of a capacitor of oscillators from the others. Using computer simulations, we observe that the van der Pol oscillators with frequency error can produce synchronization phenomena such as oscillation death, independent oscillation and double-mode oscillation for large-scale oscillatory systems. Furthermore, we investigate four types of arrangements of the van der Pol oscillators with frequency error and investigate the synchronization phenomena for each coupling method.

2. Circuit Model

The basic circuit model and the concept of circuit system are shown in Fig. 1. In the system, only the Nth oscillator has different oscillation frequency from the others. We realize the frequency error of the Nth van der Pol oscillator by setting the different value of the capacitor of this oscillator in comparison to the others.



(a) Ring of van der Pol oscillators.



(b) Conceptual circuit model.

Figure 1: Circuit model and concept of circuit system.

First, we assume that the $v_k - i_{Rk}$ characteristics of the nonlinear resistor in each oscillator is represented by the following third order polynomial equation.

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3 \tag{1}$$

By changing the variables and the parameters,

$$v_k = \sqrt{\frac{g_1}{3g_3}} x_k, \quad i_k = \sqrt{\frac{g_1}{3g_3}} \sqrt{\frac{C}{L}} y_k, \quad t = \sqrt{LC}\tau,$$

$$\varepsilon = g_1 \sqrt{\frac{L}{C}}, \quad \gamma = r\sqrt{\frac{C}{L}}, \quad \alpha = \frac{1}{\omega^2}, \quad \eta = r_m \sqrt{\frac{C}{L}},$$

the normalized circuit equations of the ring of oscillators are given as

$$\begin{cases} \dfrac{dx_k}{d\tau} = \omega_k^2 \varepsilon x_k(1 - {x_N}^2) - \omega_k^2 (y_{aN} + y_{bN}) \\ \dfrac{dy_{ak}}{d\tau} = \dfrac{1}{2} x_k - \eta y_{ak} - \gamma(y_{ak)} + y_{b(k+1)}) \\ \dfrac{dy_{bk}}{d\tau} = \dfrac{1}{2} x_k - \eta y_{bk} - \gamma(y_{a(k-1)} + y_{bk)}) \\ \qquad\qquad\qquad (k = 1, 2, \cdots, N) \end{cases} \tag{2}$$

where

$$y_{a0} = y_{aN}, \qquad y_{b(N+1)} = y_{b1}. \tag{3}$$

It should be noted that ω_k denotes the frequency of kth oscillators, γ corresponds to the coupling strength and that ε corresponds to the nonlinearity of oscillators. Eq. (2) is calculated by using the fourth-order Runge-Kutta method.

3. Synchronization Phenomena

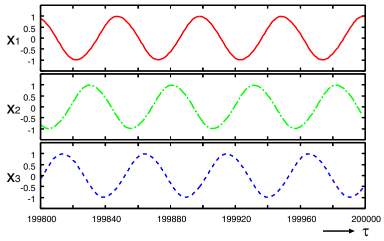
Figure 2 shows the observed phenomena for coupling oscillator number N=3. For the case that all of these oscillations have the same frequency, we can observe that the system is synchronized at the three-phase (Fig. 2(a)). When the frequency of the 3rd oscillator is varied, we confirm that oscillation of the 3rd oscillator stops, namely oscillation death appears (Fig. 2(b)). As increasing the frequency of the 3rd oscillator, oscillation of the 3rd oscillator starts again (Fig. 2(c)). However, the 3rd oscillator is not synchronized to the others. Namely, the 3rd oscillator oscillates alone and the others keep anti-phase synchronization. Furthermore, we observe the double-mode oscillation as shown in Fig. 2(d).

Next, we calculate the relationship between the amplitude of the Nth oscillator and the frequency ω_N when the nonlinearity parameter are changed from ε=0.06 to 0.2. The simulated results for N=3, 4, 9, 10, 15 and 16 are shown in Fig. 3. The graph form of amplitude is different between odd and even number of coupling oscillators. In the case of odd number, the amplitude takes the peak on ω_N=1.0 and around ω_N=1.2 the oscillation death appears when the nonlinearity parameter ε_N is set to 0.06. While, in the case of even number, although the amplitude takes the peak on ω=1.0, the oscillation death can not be observed.

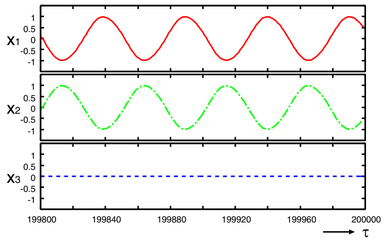
Figure 4 shows the frequency at the break down of the anti-phase or N-phase synchronization by changing the number of oscillators. From these results, we confirm that the even number coupling is more stable than the odd number coupling systems for the frequency error.

4. Group Coupling

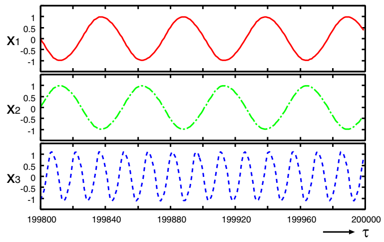
In this section, we investigate four types of arrangements of the van der Pol oscillators with frequency error and investigate the synchronization phenomena for each coupling method.



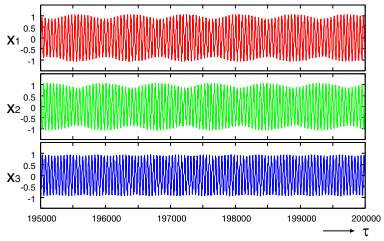
(a) Three-phase synchronization (ω_1=ω_3=ω_3=1.0).



(b) Anti-phase and oscillation death (ω_1=1.0, ω_2=1.0, ω_3=0.64).



(c) Anti-phase and independent oscillation (ω_1=1.0, ω_2=1.0, ω_3=3.47).



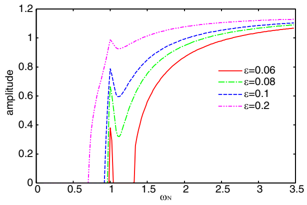
(d) Double-mode oscillation (ω_1=1.0, ω_2=1.0, ω_3=1.07).

Figure 2: Computer calculation for N=3 (ε = 0.2, γ = 0.2, η = 0.01).

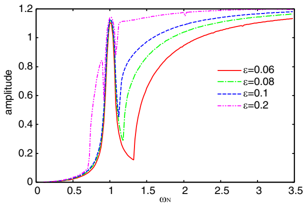
4.1. Two groups

First, we consider that the coupled oscillatory systems are composed of two groups with different oscillatory frequencies as shown in Fig. 5. In the Type-A, the two types of oscillators with different frequencies are set to half and half. And, in the case of Type-B, we place the two types of oscillators alternately. From Fig. 5, α denotes the different frequency to the standard oscillators with ω=1.0. As a first step, we consider the case of that number of coupling oscillators are set to even number to distinguish two groups clearly.

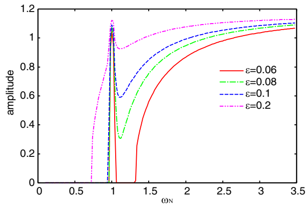
The simulated results of Type-A are shown in Fig. 6. From this figure, we confirm that each group are synchronized in anti-phase with own frequency and two groups do



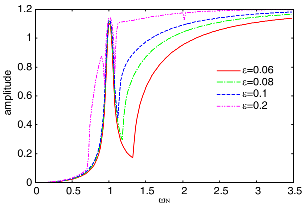
(a) N=3.



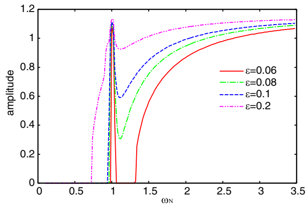
(b) N=4.



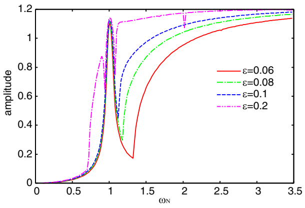
(c) N=9.



(d) N=10.

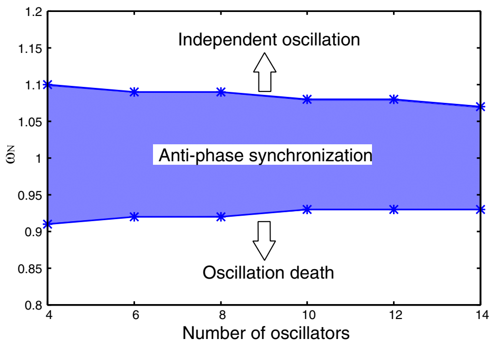


(e) N=15.

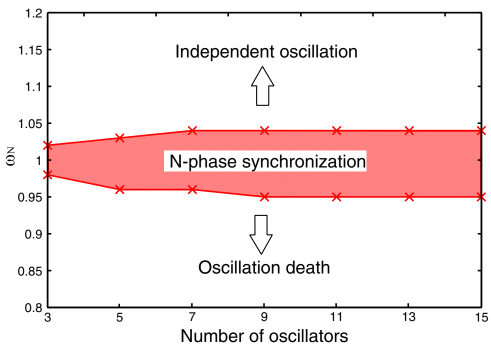


(f) N=16.

Figure 3: Relation between amplitude and ω_N.



(a) Break down for anti-phase synchronization.

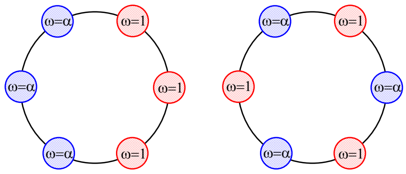


(b) Break down for N-phase synchronization.

Figure 4: Frequency for break down of synchronization ($\varepsilon = 0.2, \gamma = 0.2, \eta = 0.01$).

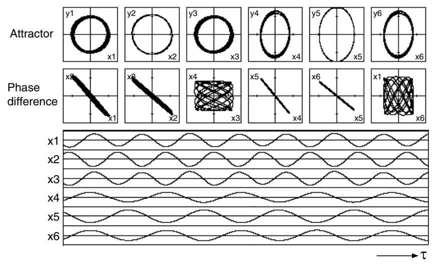
not synchronize. Figure 7 shows the observed phenomena of coupled oscillators as Type-B. In this case, every oscillators are not synchronized. When the value of α equal to 0.64, the oscillation death can be observed (Fig. 7(a)).

Next, we compare the break down frequency from synchronization to non-synchronization between Type-A and Type-B coupling methods by changing the number of coupling oscillators from 4 to 14. Where "non-synchronization" denotes the break down of group synchronization. Figure 8 shows the frequency for break down of synchronization. In the case of Type-A, the synchronization area is decreasing with number of coupling oscillators. While in the case of Type-B, the frequency at break down of synchronization is constant even if the number of coupling oscillators is changed. We confirm that Type-B is more stable for synchronization by increasing the frequency error than Type-A coupling.

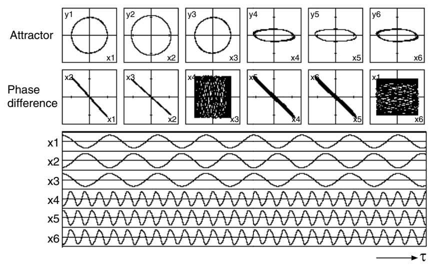


(a) Type-A (b) Type-B

Figure 5: Coupling model for two groups.



(a) α=0.64.

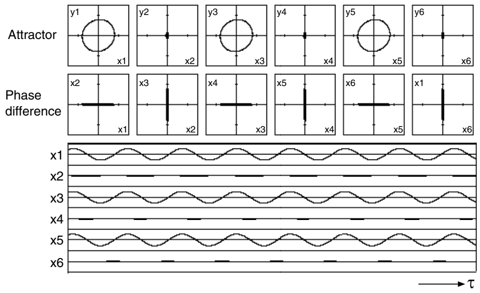


(b) α=3.47.

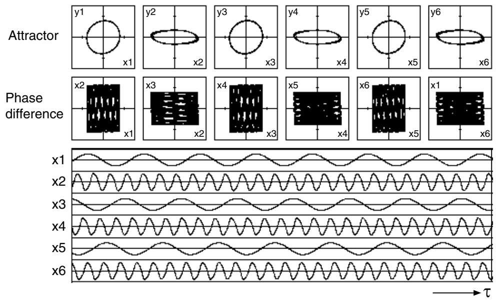
Figure 6: Computer simulation results of Type-A (N=6).

4.2. Three groups

Next, we consider that the coupled oscillatory systems are composed of three groups with different oscillatory fre-

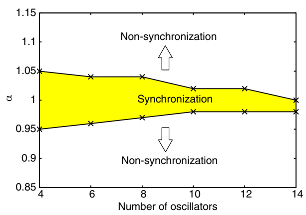


(a) α=0.64.

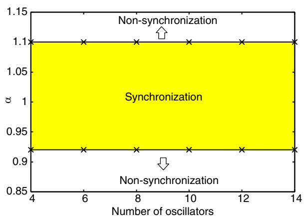


(b) α=3.47.

Figure 7: Computer simulation results of Type-B (N=6).

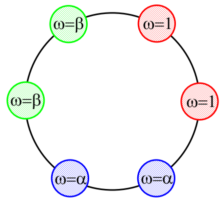


(a) Type-A.

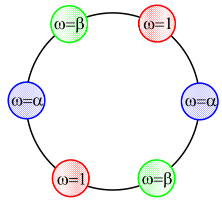


(b) Type-B.

Figure 8: Frequency for break down of synchronization (ε = 0.2, γ = 0.2, η = 0.01).

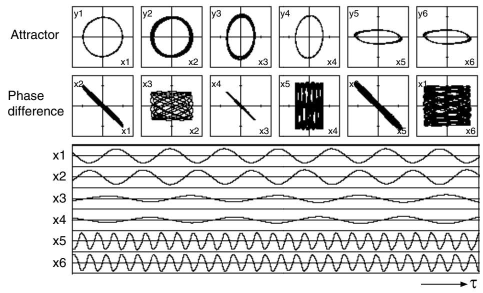


(a) Type-C

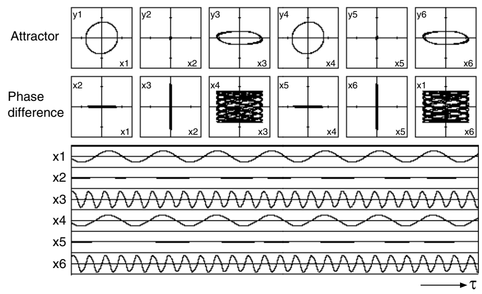


(b) Type-D

Figure 9: Coupling model for three groups.



(a) Type-A.



(b) Type-B.

Figure 10: Computer simulation results of Type-C and Type-D (N=6, α=0.64, β=3.47).

quencies as shown in Fig. 9. In the Type-C, the three types of oscillators with different frequencies are set as shown in Fig. 9(a). And, in the case of Type-D, we place the three types of oscillators alternately. From Fig. 9, α and β denote the different frequencies to the standard oscillators with ω=1.0.

The simulated results are shown in Fig. 10. From this figure, in the case of Type-C, we confirm that each group are synchronized in anti-phase with own frequency and two groups do not synchronize. In the Type-D, any adjacent oscillators are not synchronized.

5. Conclusions

In this study, we have investigated a ring coupling van der Pol oscillators with different oscillation frequencies. By computer simulations, we observe that the van der Pol oscillators with frequency error can produce the interesting nonlinear phenomena such as oscillation death, independent oscillation and double-mode oscillation.

References

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