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Nonlinear Spring Model of Self-Organizing Map Arranged in Two-Dimensional Array

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Abstract—In this study, we propose a nonlinear spring model of Self-Organizing Map (SOM) arranged in 2-dimensional grid. The neurons of the proposed model consists of $N \times M$ neurons located at a rectangular grid and are connected by the nonlinear spring. We investigate the chaotic behaviors of the model connected in 2×2 and 5×5.

1. Introduction

The Self-Organizing Map (SOM) is a subtype of artificial neural networks [1]. It is trained using unsupervised learning and is a model simplifying self-organization process of the brain. However, SOM is still far away from the realization of the brain mechanism. In order to realize more powerful and more flexible mechanism, it is important to propose new models of the brain mechanism and to investigate their behaviors.

In our previous research, as the first step to realize a new nonlinear spring model of SOM, we have proposed a simple one dimensional 2 and 3-neuron model connected by a nonlinear spring [2], [3]. We have investigated its behavior under a simple assumption where input vectors are given to the model periodically.

In this study, since SOM arranged in 2-dimension is the most well-used for applications of SOM, we propose $N \times M$ SOM model connected by the nonlinear spring. The neurons of the proposed model consists of $N \times M$ neurons located at a rectangular grid. In the SOM algorithm, the neuron nearer to the winner can be updated more significantly. We represent a relationship between the winner and its neighboring neurons. The input vectors are given to the 4 corners of the model, and the neuron nearest to the input becomes a winner and is attracted to the input vector. The other neurons always do not receive the direct effect from the input vector and are influenced only by the restoring force of the nonlinear spring from the neighboring neurons. We investigate the chaotic behaviors of the model connected in 2×2 and 5×5.

2. Nonlinear Spring Model of SOM Arranged in $N \times M$

In this study, we propose the nonlinear spring model of SOM with a rectangular structure. The proposed model consists of neurons located at $N \times M$ rectangular grid. The model is shown in Fig. 1. All the neurons are assumed to have same mass m and to be connected by the nonlinear spring with the natural length l whose restoring force F against the variation x is represented by $F = -bx^3$ where b denotes the stiffness of the spring.

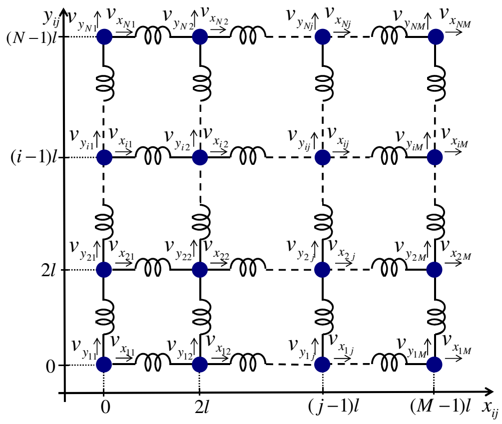

Figure 1: Nonlinear spring model of SOM with $N \times M$ rectangular shape.

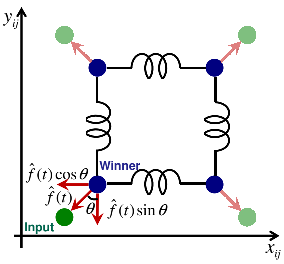

Figure 2: Input vector and winner.

Without loss of generality, we fix the position of the Neuron 11, which is located at 1st low and 1st column, as the origin of the x and y-coordinate. The state variables are the positions of other neurons denoted by x_{ij} and y_{ij} and the velocities of the neurons denoted by $\hat{v}_{x_{ij}}$ and $\hat{v}_{y_{ij}}$.

Furthermore, we model the learning process of the SOM by the external input vectors. In the SOM algorithm, a neuron nearest to the input vector becomes a winner, and it and its neighboring neurons are attracted to the input vector. In this study, we concentrate on the case that the input vectors are given to the 4 neurons of the model. Therefore, the neuron nearest to the input, namely the Neuron 11, $1M$, $N1$ or

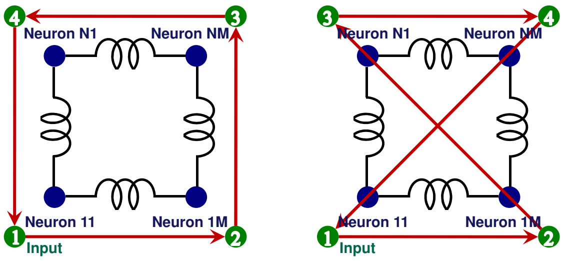

Figure 3: Input patterns. (a) *Rotation A*; Neuron 11, $1M$, NM and $N1$. (b) *Rotation B*; Neuron 11, $1M$, $N1$ and NM.

NM, becomes the winner as Fig. 2, and it is attracted to the input. Note that the other neurons do not receive a direct effect from the input vector. We consider two kinds of input patterns. One case is that the input vectors are given to the corner of the model near the Neuron 11, $1M$, NM and $N1$ in rotation; called *Rotation A* as Fig. 3(a). Another case is that they are given to the corner near Neuron 11, $1M$, $N1$ and NM in rotation; called *Rotation B* as Fig. 3(b). In case of *Rotation A*, the motion equation is described according to the position of the neuron. For the all neurons,

$$\left\{\begin{array}{l} \dot{x}_{ij} = v_{x_{ij}} \\ \dot{y}_{ij} = v_{y_{ij}} \end{array}\right. \tag{1}$$

For 4 corner neurons ; Neuron 11 as

$$\left\{\begin{array}{l} \dot{v}_{x_{11}} = -kv_{x_{11}} + Fx_{11,12} + Fx_{11,21} - f(\tau)\cos\theta \\ \dot{v}_{y_{11}} = -kv_{y_{11}} + Fy_{11,12} + Fy_{11,21} - f(\tau)\sin\theta, \end{array}\right. \tag{2}$$

Neuron $1M$ as

$$\left\{\begin{array}{l} \dot{v}_{x_{1M}} = -kv_{x_{1M}} - Fx_{1(M-1),1M} + Fx_{1M,2m} \\ \qquad\qquad + f(\tau - \frac{\pi}{2})\cos\theta \\ \dot{v}_{y_{1M}} = -kv_{y_{1M}} - Fy_{1(M-1),1M} + Fy_{1M,2m} \\ \qquad\qquad - f(\tau - \frac{\pi}{2})\sin\theta, \end{array}\right. \tag{3}$$

Neuron NM as

$$\left\{\begin{array}{l} \dot{v}_{x_{NM}} = -kv_{x_{NM}} - Fx_{N(M-1),NM} - Fx_{(N-1)M,NM} \\ \qquad\qquad + f(\tau - \pi)\cos\theta \\ \dot{v}_{y_{NM}} = -kv_{y_{NM}} - Fy_{N(M-1),NM} - Fy_{(N-1)M,NM} \\ \qquad\qquad + f(\tau - \pi)\sin\theta, \end{array}\right. \tag{4}$$

Neuron $N1$ as

$$\left\{\begin{array}{l} \dot{v}_{x_{N1}} = -kv_{x_{N1}} + Fx_{N1,n2} - Fx_{(N-1)1,N1} \\ \qquad\qquad - f(\tau - \frac{3}{2}\pi)\cos\theta \\ \dot{v}_{y_{N1}} = -kv_{y_{N1}} + Fy_{N1,n2} - Fy_{(N-1)1,N1} \\ \qquad\qquad + f(\tau - \frac{3}{2}\pi)\sin\theta. \end{array}\right. \tag{5}$$

For the neurons at the bottom row,

$$\left\{\begin{array}{l} \dot{v}_{x_{1j}} = -kv_{x_{1j}} - Fx_{1(j-1),1j} + Fx_{1j,1(j+1)} + Fx_{1j,2j} \\ \dot{v}_{y_{1j}} = -kv_{y_{1j}} - Fy_{1(j-1),1j} + Fy_{1j,1(j+1)} + Fy_{1j,2j}. \end{array}\right. \tag{6}$$

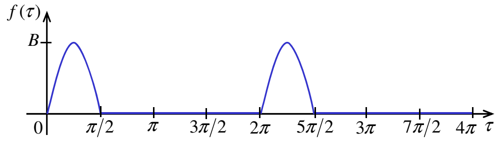

Figure 4: Force by external input vector.

For neurons at the left column,

$$\left\{\begin{array}{l} \dot{v}_{x_{i1}} = -kv_{x_{i1}} + Fx_{i1,i2} - Fx_{(i-1)1,i1} + Fx_{i1,(i+1)1} \\ \dot{v}_{y_{i1}} = -kv_{y_{i1}} + Fy_{i1,i2} - Fy_{(i-1)1,i1} + Fy_{i1,(i+1)1}. \end{array}\right. \tag{7}$$

For neurons at the right column,

$$\left\{\begin{array}{l} \dot{v}_{x_{iM}} = -kv_{x_{iM}} - Fx_{i(N-1),iM} - Fx_{(i-1)N,iM} + Fx_{iM,(i+1)N} \\ \dot{v}_{y_{iM}} = -kv_{y_{iM}} - Fy_{i(N-1),iM} - Fy_{(i-1)N,iM} + Fy_{iM,(i+1)N}. \end{array}\right. \tag{8}$$

For neurons at the top row,

$$\left\{\begin{array}{l} \dot{v}_{x_{Nj}} = -kv_{x_{Nj}} - Fx_{N(j-1),Nj} + Fx_{Nj,N(j+1)} - Fx_{(N-1)j,Nj} \\ \dot{v}_{y_{Nj}} = -kv_{y_{Nj}} - Fy_{N(j-1),Nj} + Fy_{Nj,N(j+1)} - Fy_{(N-1)j,Nj}. \end{array}\right. \tag{9}$$

For the other neurons,

$$\left\{\begin{array}{l} \dot{v}_{x_{ij}} = -kv_{x_{ij}} - Fx_{ij,i(j-1)} + Fx_{i(j+1),ij} \\ \qquad\qquad - Fx_{ij,(i-1)j} + Fx_{(i+1)j,ij} \\ \dot{v}_{y_{ij}} = -kv_{y_{ij}} - Fy_{ij,i(j-1)} + Fy_{i(j+1),ij} \\ \qquad\qquad - Fy_{ij,(i-1)j} + Fy_{(i+1)j,ij}, \end{array}\right. \tag{10}$$

where $(i = 1, 2, \cdots, N)$, $(j = 1, 2, \cdots, M)$, $x_{11} = 0$, $y_{11} = 0$ and $\theta = \pi/4$. We use following normalization parameters;

$$\begin{array}{l} \text{“}\cdot\text{”} = \frac{d}{d\tau}, \quad \hat{v}_{x_{ij}} = \sqrt{\frac{b}{m}}\, v_{x_{ij}}, \quad \hat{v}_{y_{ij}} = \sqrt{\frac{b}{m}}\, v_{y_{ij}}, \\ t = \sqrt{\frac{m}{b}}\,\tau, \quad k = \frac{a}{\sqrt{bm}}, \end{array} \tag{11}$$

where a is the friction parameter. $Fx_{ij,\hat{i}\hat{j}}$ and $Fy_{ij,\hat{i}\hat{j}}$ are x and y-component of the spring forces between the Neuron ij and $\hat{i}\hat{j}$ as following equations;

$$\left\{\begin{array}{l} Fx_{ij,\hat{i}\hat{j}} = \left(\sqrt{((x_{\hat{i}\hat{j}} - x_{ij})^2 + (y_{\hat{i}\hat{j}} - y_{ij})^2} - l\right)^3 \\ \qquad\qquad \times \frac{x_{\hat{i}\hat{j}} - x_{ij}}{\sqrt{((x_{\hat{i}\hat{j}} - x_{ij})^2 + (y_{\hat{i}\hat{j}} - y_{ij})^2}} \\ Fy_{ij,\hat{i}\hat{j}} = \left(\sqrt{((x_{\hat{i}\hat{j}} - x_{ij})^2 + (y_{\hat{i}\hat{j}} - y_{ij})^2} - l\right)^3 \\ \qquad\qquad \times \frac{y_{\hat{i}\hat{j}} - y_{ij}}{\sqrt{((x_{\hat{i}\hat{j}} - x_{ij})^2 + (y_{\hat{i}\hat{j}} - y_{ij})^2}}. \end{array}\right. \tag{12}$$

$f(\tau)$ is the force by the external input vectors as

$$f(\tau) = \left\{\begin{array}{ll} B\sin(2\tau), & 2n\pi \le \tau \le \left(2n\pi + \frac{\pi}{2}\right) \\ & (n = 0, 1, \cdots) \\ 0, & \text{otherwise} \end{array}\right. \tag{13}$$

where $t = 0$ is the time when the input vector is given. Because Eqs. (2)–(5) are for the case of *Rotation A*, we need to change the phases for the other input pattern. The

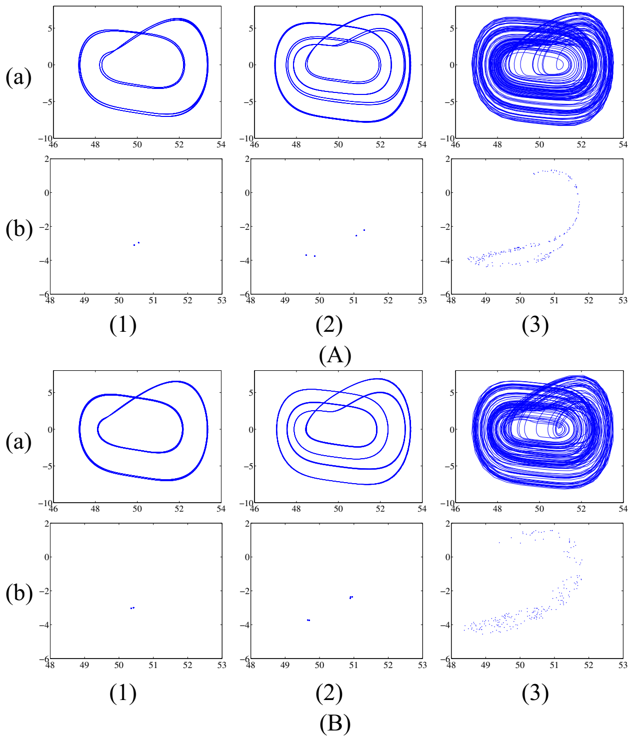

Figure 5: Projection of attractors onto x_{12}–$v_{x_{12}}$ plane and their Poincaré maps of 2×2 model. $B = 15$. $l = 50$. (A) *Rotation A*. (B) *Rotation B*. (a) Attractors. (b) Poincaré maps. (1) $k = 0.45$. (2) $k = 0.38$. (3) $k = 0.31$.

input vectors are given to 4 corners at fixed angle of $\pi/4$. Therefore, one of 4 neurons nearest to the input becomes the winner and is attracted to the input with the force as Eq. (13). The other neurons always do not receive the direct effect from the input vector and are influenced only by the restoring force of the nonlinear spring from the neighboring neurons. The shape of $f(\tau)$ is shown in Fig. 4.

3. Computer Simulation Results

We show some computer calculation results obtained by using Runge-Kutta method with time step $\delta t = 2\pi/500$ for Eq. (1)-(10). We define the Poincaré section as $\tau = 2n\pi$. We set the initial states of the positions to its own physical location on the 2-D grid as $x_{ij}(0) = (j-1)l$ and $y_{ij}(0) = (i-1)l$. The initial states of the velocities are set to 0.

3.1. 2×2 Model

First, we consider the 2×2 model ($N = M = 2$) with the two cases of the input methods. The projections of attractors onto x_{12}–$v_{x_{12}}$ and their Poincaré maps for both cases are shown in Figs. 5(A) and (B), respectively. As Fig. 5(A), two-periodic orbit (1) bifurcates to four-periodic orbit (2) and chaos (3). By decreasing k, chaos grows more complex. Meanwhile, in the case of *Rotation B* as Fig. 5(B), we can observe similar phenomena for the same parameters. These results mean the property of the primary SOM which the learning results are hardly influenced by changing the order of giving the input vectors.

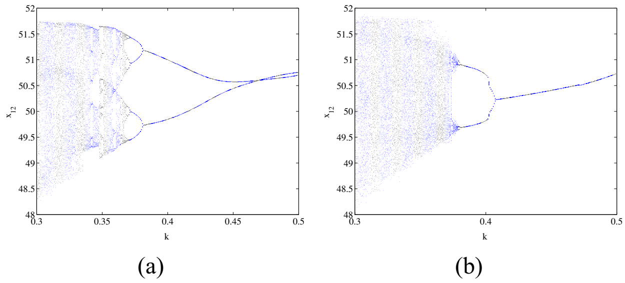

Figure 6: Bifurcation diagrams of model with 2×2 neurons for fixed $B = 15$ and $l = 50$ by changing k from 0.3 to 0.5. (a) *Rotation A*. (b) *Rotation B*.

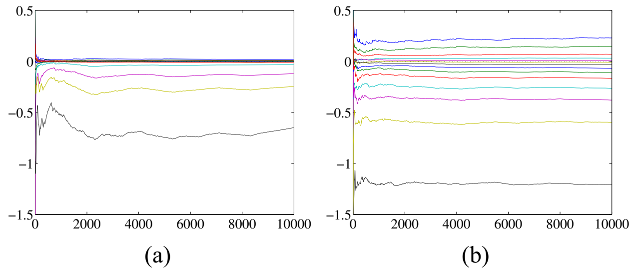

Figure 7: Lyapunov exponents. $B = 15$, $l = 50$, $k = 0.36$. (a) *Rotation A*. (b) *Rotation B*.

Moreover, we made one-parameter bifurcation diagram of the model with 4-neuron by increasing k gradually for fixed $B = 15$ and $l = 50$. Figures 6(a) and (b) show the cases of the input vector given by *Rotation A* and *Rotation B*, respectively. By changing k from 0.3 to 0.5 as Fig. 6(a), we can confirm the widely chaos region, and it is clear that the chaotic behavior becomes weak for larger k value in both cases. From Fig. 6(a), we can observe that the chaos region for $0.3 < k < 0.335$, some periodic windows and the period-doubling bifurcation correspond to Figs. 5(A1) and (A2). On the other hand, as Fig. 6(b), somewhat different bifurcations are observed in the case that input vectors are given by *Rotation B*. It means that more complex chaos than by *Rotation A* is generated by *Rotation B*.

Lyapunov exponents for case of *Rotation A* and *Rotation B* are shown in Fig. 7. Five Lyapunov exponents take positive values in both cases, namely, the neurons oscillate chaotically. The largest Lyapunov exponent is $\lambda_1 = 0.020$ and 0.229 for *Rotation A* and *Rotation B*, respectively. Therefore, the nonlinear spring model by *Rotation B* can be said to generate more complex chaos than *Rotation A*.

3.2. 5×5 Model

Next, we consider the nonlinear spring model with 5×5 neurons ($N = M = 5$) in the case of the input vectors are given by *Rotation A*. We investigate the chaotic behavior of the model with taking notice of the relationship between

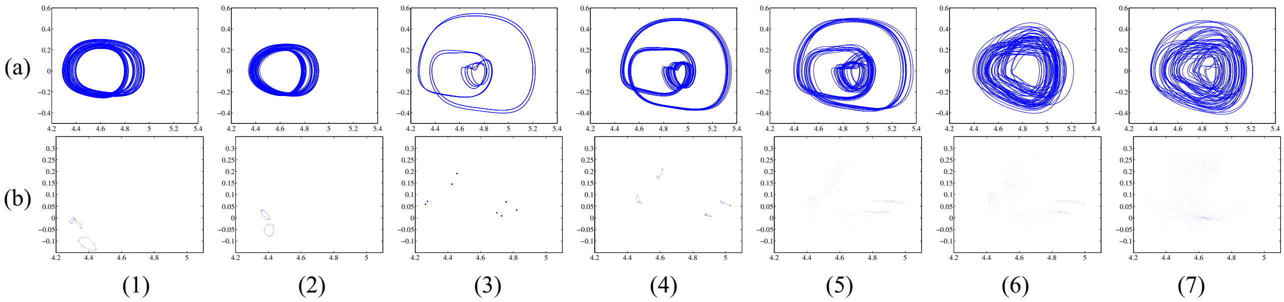

Figure 10: Projection of attractors onto x_{53}–$v_{x_{53}}$ plane and their Poincaré maps of 5×5 model. Fixed parameters are $k = 0.15$ and $l = 2$. Input vectors are given by *Rotation A*. (a) Attractors. (b) Poincaré maps. (1) $B = 0.7$. (2) $B = 0.75$. (3) $B = 0.885$. (4) $B = 0.8864$. (5) $B = 0.887$. (6) $B = 0.888$. (7) $B = 0.89$.

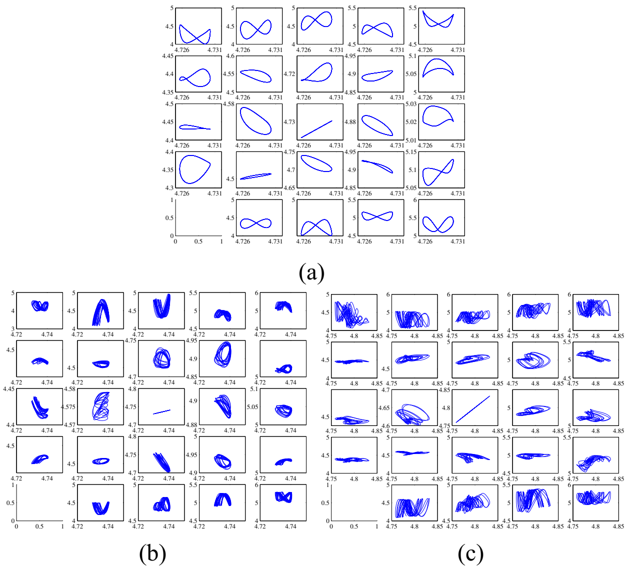

Figure 8: Attractors onto x_{33} vs $(x_{ij} - (j-3)l)$ for $k = 0.15$ and $l = 2$. (a) $B = 0.6$. (b) $B = 0.7$. (c) $B = 0.9$.

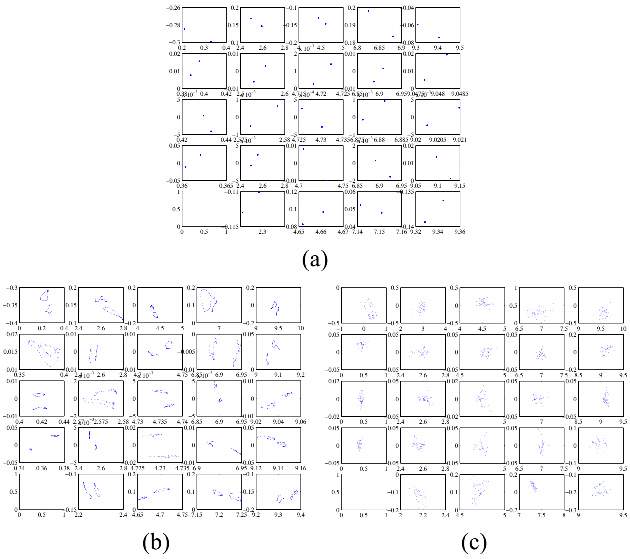

Figure 9: Poincaré maps onto x_{ij}–$v_{x_{ij}}$ plane for $k = 0.15$ and $l = 2$. (a) $B = 0.6$. (b) $B = 0.7$. (c) $B = 0.9$.

neurons. Figure 8 shows the projection of attractors of respective positions as x_{ij} based on the position of the center Neuron 33 as x_{33}. For $B = 0.6$ as Fig. 8(a), the neurons are oscillating similarly based on the Neuron 33. In fact, the attractors of 1-neighbor of the Neuron 33, namely, onto x_{33}–x_{32}, x_{33}–x_{23}, x_{33}–x_{34} and x_{33}–x_{43} plane, are similar shapes. By increasing B as Figs. 8(b) and (c), the respective neurons make different behaviors. For $B = 0.9$, we can observe that the symmetries of behaviors are broken completely and all the neurons behave chaotically.

Figure 9 shows the Poincaré maps of the respective neurons onto x_{ij}–$v_{x_{ij}}$ plane. Two-periodic orbit (a) bifurcates to torus (b). As B increases, the chaos (c) is observed.

In order to investigate the growth process from the torus to the chaos, we observe the behaviors changing B more closely as shown in Fig. 10. We can see the torus as (1) and (2). At $B = 0.885$, 8-periodic state (3) appears and the folded torus appear as (4). As B increases further, we can observe the torus breakdown. Poincaré map becomes thick and chaos generations are confirmed visually (5)-(7).

4. Conclusions

We have proposed the SOM model whose neurons are arranged in 2-dimensional array and connected by the nonlinear spring. We have obtained the similar behaviors of the model connected in 2×2 for difference input patterns. Furthermore, we have considered the model arranged 5×5 array and have investigated the chaotic behaviors with taking notice of the relationship between neurons.

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