2005 RISP International Workshop on Nonlinear
Circuit and Signal Processing (NCSP'05)
Hawaii, USA, Mar. 4-6, 2005.

# Tracing Bifurcation Branches of High-Dimensional Nonlinear Algebraic Equations Using SPICE

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**abstract**: It is very important to calculate the multiple solutions of nonlinear equations, because there are many kind of practical problems in nonlinear electronic circuits and the nonlinear equations in many scientific fields. Therefore, there have been already proposed many algorithms for calculating multiple and/or all the solutions.

One of the powerful methods is a Newton homotopy technique which can find out the multiple and/or all the solutions lying on the homotopy paths in the global space. Unfortunately, it cannot trace all the solutions when the homotopy path has bifurcation points on it, because the path happens to have separated branches at the point. In this paper, we propose an efficient algorithm to find the bifucation points at which a rank of the Jacobian matrix is reduced more than one. After finding the points, we fine the directions of branches, and trace them with the arc-length method. Our algorithm is based on the use of Spice and needs not to derive the Jacobian matrix, so that it can be easily applied to many kinds of nonlinear equations.

## 1. Introduction

Numerical analysis of nonlinear algebraic equations is one of the most fundamental and important problems in scientific and engineering fields. The problems in circuit designs are corresponding to find the multiple DC operating points, the oscillator designs, and so on. Although in Spice simulator, Newton-Raphson method is mainly used for this purpose, it can be only applied when the approximate solutions are given. Thus, it cannot any more use to find the multiple and/or all the solutions. Therefore, there have been published many papers [1]-[5] for finding the multiple and/or all the solutions of the following nonlinear equation:

$$\left.\begin{array}{c} f_1(v_1, v_2, \cdots, v_n) = 0 \\ f_2(v_1, v_2, \cdots, v_n) = 0 \\ \cdots\cdots\cdots \\ f_n(v_1, v_2, \cdots, v_n) = 0 \end{array}\right\} \quad (1)$$

Some of them [2]-[3] can efficiently find all the solutions of piecewise-linear resistive circuits, where the nonlinear elements are modeled by the piecewise-linear segments. Although the interval method [4] was proposed as a method for finding all the solutions, it is a rather time-consuming for large scale systems. On the other hand, the homotopy methods [5]-[10] have global convergence property, so that they can be applied to wide classes of nonlinear equations. Especially, they are applied to calculate the multiple solutions of nonlinear equations. The Newton-homotopy method [7] coupled with the solution curve tracing algorithm [10] is possible to trace all the solutions existing on the homotopy paths which can start from any initial value $\mathbf{v}_0$. However, it is not known whether it has found all the solutions or not, because the homotopy paths may consist of the multiple independent branches and the closed loops in the space. For an example, many branches cross together at the points called pitchfork bifurcation points. Therefore, it is important to calculate the bifurcation points to find the multiple solutions, and to know the behaviors of the nonlinear eq.(1).

In this paper, we show the curve tracing algorithm using SPICE in section 2. The Spice-oriented algorithms to find the pitchfork bifurcation points, and tracing the branches starting at the points are shown in section 3 and 4.

## 2. Curve tracing algorithm

Nowadays, Spice is widely used for circuit simulations of integrated circuits. Since nonlinear devices in ICs are modeled by many kinds of the functions, Spice has all kinds of controlled sources, and nonlinear functions such as exponential functions, multiplications, additions and subtractions and so on. Furthermore, we can easily construct abitrary functions using ABMs(analog behavior models). An example of multiple-inputs and single-output function is shown in Fig.1, where the function $f(v_1, v_2, \ldots, v_n)$ in the black box is realized by writing the function with the input variables $\{v_1, v_2, \ldots, v_n\}$. On the other hand, a set of the nonlinear equations given by eq.(1) is realized by the equivalent circuit as shown in Fig.2.

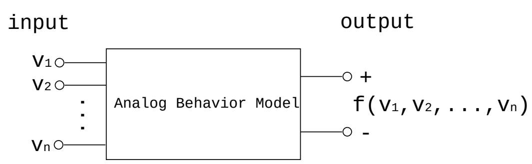



Figure 1: Analog Behavior Model.

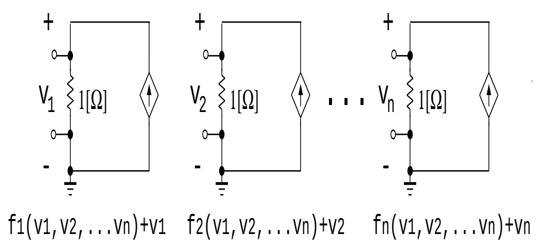



Figure 2: Equivalence circuit of the nonlinear algebraic equation.

Namely, if we assume the voltage at $i$-th resistor ($1\Omega$) by $v_i$, we can realize the current source as follows:

$$I_i = f_i(v_1, v_2, \ldots, v_n) - v_i \quad (i = 1, 2, \ldots, n) \qquad (2)$$

Thus, we have a relation $f_i(v_1, v_2, \ldots, v_n) = 0$ with the Kirchhoff's voltage law. Let us apply the *Newton homotopy method* to solve the eq.(1). Then, we formulate the following modified relation:

$$\mathbf{F}(\mathbf{v}, \rho) = \mathbf{f}(\mathbf{v}) + (\rho - 1)\mathbf{I}_0 = \mathbf{0} \qquad (3)$$

where $f(\mathbf{v}) : \mathbf{R}^n \mapsto \mathbf{R}^n$ given by the relation eq.(1), and $\mathbf{I}_0 = \mathbf{f}(\mathbf{v}_0)$ is the initial guess. Thus, the relation eq.(2) is consists "$n$" equations having "$n+1$" variables, so that the solution satisfying the relation eq.(3) is given by the solution curves in "$(n+1)$" dimensional space. In order to trace the solution curve with the arc-length method, we apply the following relation [10]:

$$\sum_{i=1}^{K} \left( \frac{dv_i}{ds} \right)^2 = 1 \qquad (4)$$

Observe that the solutions satisfying eq.(1) are obtained at $\rho = 1$ on the solution curve of eq.(3). Now, we show the equivalent circuit satisfying the Newton homotopy method eq.(3) and eq.(4). The block diagram is shown by Fig.3(a). Some of the "$K$" node voltages in the eq.(1) should be chosen as the variables of the arc-length method eq.(4) [7]. Thus, the circuit realizing the arc-length method eq.(4) is shown by Fig.3(b). That is called STC(solution curve tracing circuit)[8]. In many practical problems [7], we can get all the DC solutions from a suitable initial guess $\mathbf{I}_0$ in eq.(3).

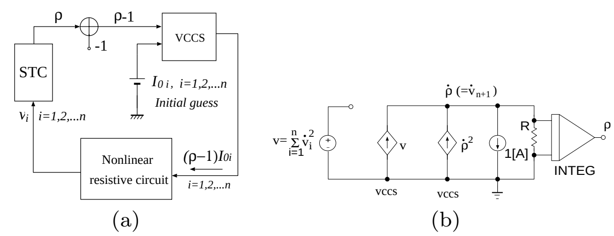



Figure 3: (a)Equivalence circuit of Newton homotopy method. (b)Solution curve tracing circuit.

## 3. Calculations of bifurcation points

Now, consider an algorithm for calculating the bifurcation points of the nonlinear equations. In this section, we show an efficient Spice-oriented algorithm. For simplicity, we set $v_{n+1} = \rho$ and rewrite the relation eq.(3) as follows:

$$\mathbf{F}(\mathbf{v}) = \mathbf{0} \qquad \mathbf{F} : \mathbf{R}^{n+1} \mapsto \mathbf{R}^n \qquad (5)$$

It is proved that our algorithm [10] can trace the solution curves only if the rank of the Jacobian matrix of eq.(5) is $n$ on the whole curve. Note that the rank at the limit points (turning points) is equal to "$n$", and the rank is less than or equal to $n-1$ at the pitchfork bifurcation points. Therefore, the limit points can be still traced by our arc-length method. However, it is generally not so easy to calculate the Jacobian matrix and the rank of eq.(5). Thus, let us to develop an algorithm to calculate the bifurcation points without estimating the Jacobian matrix.

At the pitchfork point of eq.(5), we consider a set of the following modified equations[1] :

$$\left. \begin{array}{l} \mathbf{F}_1(\mathbf{v}) = \mathbf{F}(\mathbf{v}) = \mathbf{0} \\ \mathbf{F}_2(\mathbf{v}, \Delta\mathbf{v}) = \mathbf{F}(\mathbf{v} + \Delta\mathbf{v}) - \mathbf{F}(\mathbf{v}) = \mathbf{0} \end{array} \right\} \qquad (6)$$

which have $2n+2$ variables and $2n$ equations if we assume $\Delta\mathbf{v}$ as the additional variables. Since the rank of the Jacobian matrix of $\mathbf{F}(\mathbf{v})$ is equal or less than $n-1$ [11], if we set the variables $\Delta v_1$ and $\Delta v_2$ with small fixed constants in $\mathbf{F}_2(\mathbf{v}, \Delta\mathbf{v})$), then, eq.(6) becomes the nonlinear equation with "$2n$" variables and "$2n$" equations. Hence, we can also apply the Newton homotopy method as follows:

[1] At the pitchfork points, the second order term of the Taylor expansions are assumed negligible values for the small variations $\Delta\mathbf{v}$

$$
\left.\begin{array}{l}
\mathbf{F}_1(\mathbf{v}) + (\lambda - 1)\mathbf{I}_{01} = \mathbf{0} \\
\mathbf{F}_2(\mathbf{v}, \Delta\mathbf{v}) + (\lambda - 1)\mathbf{I}_{02} = \mathbf{0} \\
\displaystyle\sum_{i=1}^{K} \left(\frac{dv_i}{ds}\right)^2 + \sum_{i=3}^{K} \left(\frac{d\Delta v_i}{ds}\right)^2 + \left(\frac{d\lambda}{ds}\right)^2 = 1
\end{array}\right\} \quad (7)
$$

Tracing the homotopy path eq.(7), we can find the bifurcation points at $\lambda = 1$. Note that two of the variables $\Delta v_1$ and $\Delta v_2$ should set to fixed sufficient small values.

**Example 3.1**. Now, let us calculate bifurcation points of the following equation:

$$
\left.\begin{array}{l}
F_1(\mathbf{v}) = (v_1^2 + v_2^2 + v_3^2 - 3)(v_1 - v_2) = 0 \\
F_2(\mathbf{v}) = v_2 - v_3 = 0
\end{array}\right\} \quad (8)
$$

We have the following modified relations:

$$
\left.\begin{array}{l}
f_1(\mathbf{v}) = (v_1^2 + v_2^2 + v_3^2 - 3)(v_1 - v_2) = 0 \\
f_2(\mathbf{v}) = v_2 - v_3 = 0 \\
f_3(\mathbf{v}, \Delta\mathbf{v}) = F_1(v_1 + \Delta v_1, v_2 + \Delta v_2, v_3 + \Delta v_3) \\
\qquad\qquad - F_1(v_1, v_2, v_3) = 0 \\
f_4(\mathbf{v}, \Delta\mathbf{v}) = F_2(v_1 + \Delta v_1, v_2 + \Delta v_2, v_3 + \Delta v_3) \\
\qquad\qquad - F_2(v_1, v_2, v_3) = 0
\end{array}\right\} \quad (9)
$$

Note that eq.(8) consists of three variables and two equation, and that it satisfies both $\frac{\partial F_1(\mathbf{v})}{\partial v_1} = 0$ and $\frac{\partial F_1(\mathbf{v})}{\partial v_2} = 0$ at the pitchfork points. Hence, the variables $\Delta v_1$, $\Delta v_2$ and $\Delta v_3$ can be set as shown in eq.(9). Applying the Newton homotopy method, we have

$$
\left.\begin{array}{l}
f_1(\mathbf{v}) + (\lambda - 1)I_{01} = 0 \\
f_2(\mathbf{v}) + (\lambda - 1)I_{02} = 0 \\
f_3(\mathbf{v}, \Delta\mathbf{v}) + (\lambda - 1)I_{03} = 0 \\
f_4(\mathbf{v}, \Delta\mathbf{v}) + (\lambda - 1)I_{04} = 0 \\
\displaystyle\sum_{i=1}^{3} \left(\frac{dv_i}{ds}\right)^2 + \left(\frac{d\Delta v_3}{ds}\right)^2 + \left(\frac{d\lambda}{ds}\right)^2 = 1
\end{array}\right\} \quad (10)
$$

where we chose $\Delta v_1$=0, $\Delta v_2$=0.01. Then, we have 2 pitchfork bifurcation points $(1.00, 1.00, 1.00)$ and $(-1.00, -1.00, -1.00)$ at $\lambda = 1$ as shown in Fig.4.

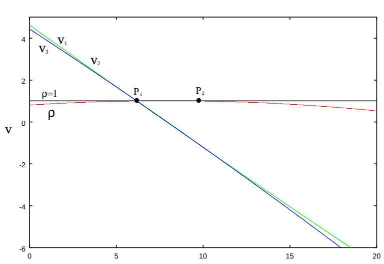



Figure 4: Solution of the pitch fork point.

The algorithm is much more simple compared to the method [12] using the Jacobian matrix of eq.(5), especially for the large scale systems.

## 4. Tracing solution curves from bifurcation points

Now, consider to calculate the branches starting from the pitchfork bifurcation points in eq.(5). We set a small sphere centered at the bifurcation points $\hat{\mathbf{v}}$, and get the intersections between the branches of eq.(5) and sphere. Thus, we have

$$
\left.\mathbf{F}(\mathbf{v}) = \mathbf{0} \quad , \quad \sum_{i=1}^{K} (v_i - \hat{v}_i)^2 = r^2 \right\} \quad (11)
$$

for a sufficient small positive value $r$. The relations eq.(11) are also solved by the Newton homotopy method. We assume that the values $(p_1, p_2, p_3, p_4)$ obtained from the eq.(11) are the initial values of the branches starting from the bifurcation point as shown in Fig.5.

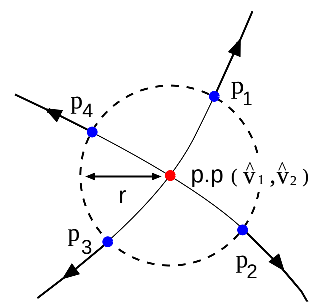



Figure 5: Tracing branches from a bifurcation point.

**Example 4.1**.
We trace the solution curve of the following equation.

$$
f(v_1, v_2) = (v_1^2 - v_2)(v_1^3 + v_1^2 - v_2^2) = 0 \quad (12)
$$

We set the fixed small variables $\Delta v_1$ and $\Delta v_2$. Thus, we have the following modified relations:

$$
\left.\begin{array}{l}
f_1(\mathbf{v}) = (v_1^2 - v_2)(v_1^3 + v_1^2 - v_2^2) = 0 \\
f_2(\mathbf{v}, \boldsymbol{\Delta}\mathbf{v}) = f(v_1 + \Delta v_1, v_2 + \Delta v_2) - f(v_1, v_2) = 0
\end{array}\right\} \quad (13)
$$

Applying the Newton homotopy method, we have

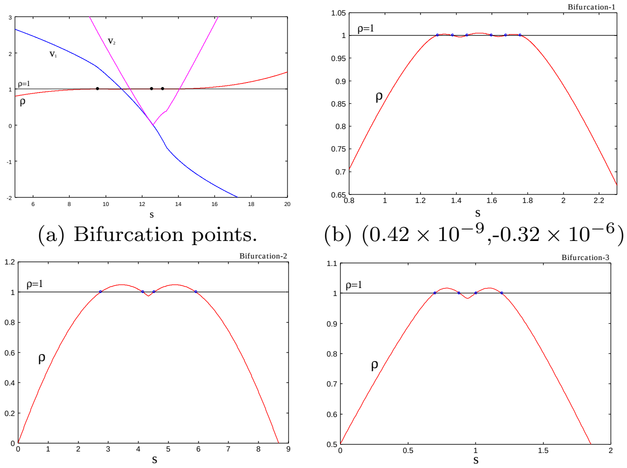
$$
\left.\begin{array}{l}
f_1(\mathbf{v}) + (\rho - 1)I_{01} = 0 \\
f_2(\mathbf{v}, \boldsymbol{\Delta}\mathbf{v}) + (\rho - 1)I_{02} = 0 \\
\displaystyle\left(\frac{dv_1}{ds}\right)^2 + \left(\frac{dv_2}{ds}\right)^2 + \left(\frac{d\rho}{ds}\right)^2 = 1
\end{array}\right\} \quad (14)
$$

where we chose $\Delta v_1$=0, $\Delta v_2$=0.01. Then, we have 3 pitchfork bifurcation points $(0.42 \times 10^{-9}, -0.32 \times 10^{-6})$, $(1.62, 2.62)$ and $(-0.62, 0.38)$ at $\lambda = 1$ as shown in Fig.6(a).

Applying the eq.(11) and Newton homotopy method, to obtain the initial value of the branchies. Thus, we obtain the following equation:

$$\left.\begin{array}{l}(v_1^2 - v_2)(v_1^3 + v_1^2 - v_2^2) + (\rho - 1)I_{03} = 0 \\ (v_1 - \hat{v}_1)^2 + (v_2 - \hat{v}_2)^2 - r^2 + (\rho - 1)I_{04} = 0 \\ \left(\dfrac{dv_1}{ds}\right)^2 + \left(\dfrac{dv_2}{ds}\right)^2 + \left(\dfrac{d\rho}{ds}\right)^2 = 1\end{array}\right\} \quad (15)$$

The results for the above example are shown by Fig.6(b),(c) and (d), respectively.




(a) Bifurcation points. (b) $(0.42 \times 10^{-9}$,$-0.32 \times 10^{-6})$

(c) (1.62,2.62) (d) (-0.62,0.38)

Figure 6: Solving of bifurcation points and initial value.

Starting from these intersections of bifurcation points, we can trace the solution curves of eq.(12) with arc-length method [9].(Fig.7)

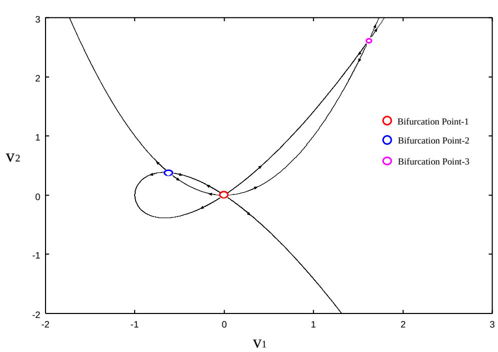



Figure 7: Solution curves of eq.(12).

## 5. Conclusions and remarks

It is very important to find all the solutions of nonlinear equations. In this paper, we proposed an efficient Spice-oriented algorithm for finding bifurcation points, where we never use the Jacobian matrices and only use the set of algebraic equations having auxiliary variables. Furthermore, we have used Newton homotopy method for finding the multiple solutions of nonlinear equations. Therefore, we can trace the solution curves with the transient algorithm of SPICE. After finding the singular points, we restart to trace the solutions curves in the different directions. We could trace all the solution curves and find all the solutions. But, although it is impossible to prove whether all the solutions of the nonlinear algebraic equations are found with our method or not, all the solutions can be found in many practical problems. In the future problems, we need to extend the algorithm to large dimensional cases.

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