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Stability Analysis of Multimode Oscillations in Two Coupled Oscillators with Ninth-Power Nonlinearities

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1. Introduction

Recently, multimode oscillations in mutually coupled oscillator systems are considerable research interest[1]. Stability of multimode oscillations in two coupled van der Pol oscillators with fifth-power nonlinearity have been investigated by Datardina and Linkens[2]. They have confirmed that nonresonant double-mode oscillations, which could not occur for the third-power nonlinearity, were stability excited in the coupled systems. In our former study, we have analyzed multimode oscillations observed from two van der Pol oscillators with ninth-power nonlinearity coupled by an inductor. By computer calculations and circuit experiments, we have confirmed that the circuit has three different types of double-mode oscillation[3].

In this study, we analyze the stability of multimode oscillations in two coupled van der Pol oscillators with ninth-power nonlinearity. The theoretical results show three different double-mode oscillations coexist for a range of parameter values. These agree well with the results of computer calculations and circuit experiments.

2. Double-mode Oscillation

Two coupled van der Pol oscillator is shown in Fig. 1. The normalized equations are given as follows:

$$\begin{aligned}&\ddot{x_1} + (1+\alpha)x_1 - \alpha x_2\\ &\quad = -\varepsilon(\dot{x_1} - \tfrac{\varepsilon_a}{3}\dot{x_1}^3 + \tfrac{\varepsilon_b}{5}\dot{x_1}^5 - \tfrac{\varepsilon_c}{7}\dot{x_1}^7 + \tfrac{1}{9}\dot{x_1}^9) = 0,\\ &\ddot{x_2} - \alpha x_1 + (1+\alpha)x_2\\ &\quad = -\varepsilon(\dot{x_2} - \tfrac{\varepsilon_a}{3}\dot{x_2}^3 + \tfrac{\varepsilon_b}{5}\dot{x_2}^5 - \tfrac{\varepsilon_c}{7}\dot{x_2}^7 + \tfrac{1}{9}\dot{x_2}^9) = 0\end{aligned} \quad (1)$$

where α is the coupling factor, ε is the degree of nonlinearity, ε_a, ε_b and ε_c define both the threshold and the amplitude of oscillation, x_1 and x_2 correspond to v_1 and v_2, respectively. Figures 2 show the computer calculated results. The parameter values are $\alpha = 0.1$, $\varepsilon = 0.3$, $\varepsilon_a = 11.29$, $\varepsilon_b = 16.91$ and $\varepsilon_c = 7.53$. Further, Equations (1) is calculated by using the Runge-Kutta method with step size $h = 0.005$. As a result, we confirm three different double-mode oscillations coexist.

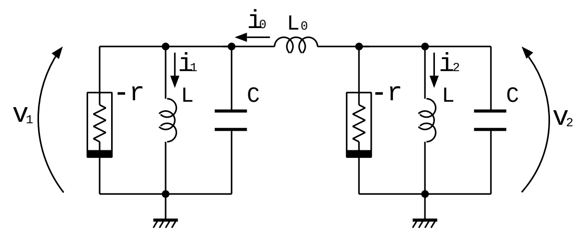


Fig. 1: Two coupled van der Pol oscillator.

3. Theoretical Analysis

Equations (1) may be more conveniently rewritten as follows:

$$\begin{aligned}&\tfrac{d^2\boldsymbol{x}}{d\tau^2} + B_N\boldsymbol{x}\\ &\quad = -\varepsilon\left(\tfrac{d\boldsymbol{x}}{d\tau} - \tfrac{\varepsilon_a}{3}\tfrac{d\boldsymbol{x}_a}{d\tau} + \tfrac{\varepsilon_b}{5}\tfrac{d\boldsymbol{x}_b}{d\tau} - \tfrac{\varepsilon_c}{7}\tfrac{d\boldsymbol{x}_c}{d\tau} + \tfrac{1}{9}\tfrac{d\boldsymbol{x}_d}{d\tau}\right)\end{aligned} \quad (2)$$

where

$$\begin{aligned}&\boldsymbol{x} = [\,x_1, x_2\,]^t, \ \ B_N = \begin{bmatrix} 1+\alpha & -\alpha \\ -\alpha & 1+\alpha \end{bmatrix},\\ &\boldsymbol{x}_a = [\,{x_1}^3, {x_2}^3\,]^t, \ \ \boldsymbol{x}_b = [\,{x_1}^5, {x_2}^5\,]^t,\\ &\boldsymbol{x}_c = [\,{x_1}^7, {x_2}^7\,]^t \ \text{ and } \ \boldsymbol{x}_d = [\,{x_1}^9, {x_2}^9\,]^t.\end{aligned} \quad (3)$$

3.1. Normalization of Matrix B_N

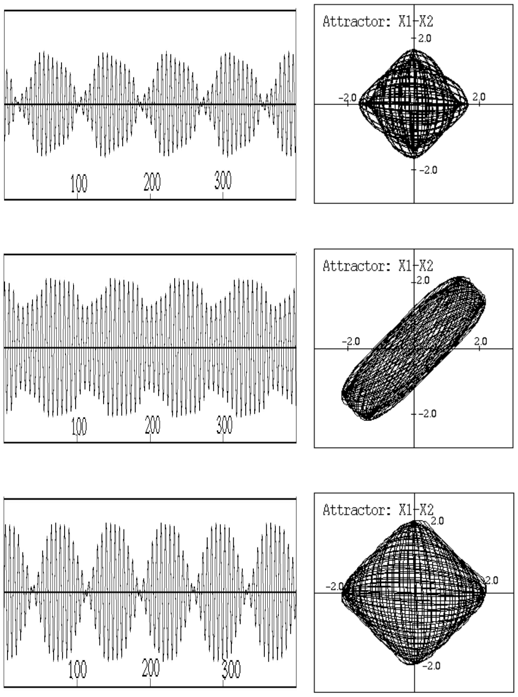
An unperturbed equation of Eq. (2):

$$\frac{d^2\boldsymbol{x}}{d\tau^2} + B_N\boldsymbol{x} = 0 \quad (4)$$

can be transformed into the following canonical form. Using nonsingular linear transformation:

$$\boldsymbol{x} = P\boldsymbol{y}, \qquad \boldsymbol{y} = [\,x_1, x_2\,]^t \quad (5)$$

$$\frac{d^2\boldsymbol{y}}{d\tau^2} + (P^{-1}B_NP)\boldsymbol{y} = 0. \quad (6)$$



(a) (b)

Fig. 2: Computer calculated results of three double-mode oscillations. (a) Time wave form. Vertical axis is τ and horizontal axis is x_1. (b) Attractor. Vertical axis is x_1 and horizontal axis is x_2.

The orthogonal matrix P may be determined uniquely by solving the difference equation, obtained by substituting Eq. (5) into Eq. (4), with a suitable boundary condition:

$$\begin{aligned} p_{ij} &= \cos\frac{(2i-1)(j-1)}{4}\pi, \quad (j \neq 1, \ i = 1,2) \\ p_{i1} &= \sqrt{\tfrac{1}{2}}, \\ \lambda_j &= 1 + \alpha - 2\alpha\cos\frac{j-1}{2}\pi, \quad (j = 1,2) \end{aligned} \tag{7}$$

where λ_j are eigenvalues and p_{ij} are eigenvectors of matrix B_N. Now, using Eq. (7) and Eq. (6) can be written

$$\frac{d^2\boldsymbol{y}}{d\tau^2} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{y} = 0 \tag{8}$$

and the solution y_i is simply determined by

$$y_i = A_i \cos(\sqrt{\lambda_i}\tau + \phi_i). \qquad (i = 1,2) \tag{9}$$

3.2. Equivalent Linearization of Nonlinear Term

In this section, we try to linearize the nonlinear terms of Eq. (2). Kryloff and Bogoliuboff linearization technique is used to linearize the ninth-power terms in Eq. (2). From Eq. (5), the term of ${x_1}^3$ can be written

$$\begin{aligned} {x_1}^3 &= (p_{11}y_1 + p_{12}y_2)^3 \\ &= {p_{11}}^3{y_1}^3 + 3{p_{11}}^2{y_1}^2 p_{12}y_2 \\ &\quad + 3p_{11}y_1{p_{12}}^2{y_2}^2 + {p_{12}}^3{y_2}^3. \end{aligned} \tag{10}$$

Substituting Eq. (9) into Eq. (10), ${x_1}^3$ can be written

$$\begin{aligned} {x_1}^3 &= {p_{11}}^3{A_1}^3\cos^3\gamma_1 + {p_{12}}^3{A_2}^3\cos^3\gamma_2 \\ &\quad + 3{p_{11}}^2 p_{12}{A_1}^2 A_2 \cos^2\gamma_1 \cos\gamma_2 \\ &\quad + 3p_{11}{p_{12}}^2 A_1{A_2}^2 \cos\gamma_1\cos^2\gamma_2 \end{aligned} \tag{11}$$

where

$$\gamma_k = \sqrt{\lambda_k}\tau_k + \phi_k. \qquad (k = 1,2) \tag{12}$$

Now, using following formula:

$$\begin{aligned} \cos^3\theta &= \tfrac{1}{4}(\cos 3\theta + 3\cos\theta) \\ \cos^2\theta &= \tfrac{1}{2}(\cos 2\theta + 1). \end{aligned} \tag{13}$$

Substituting Eq. (13) into Eq. (11), ${x_1}^3$ can be written

$$\begin{aligned} {x_1}^3 &= \tfrac{1}{4}{p_{11}}^3{A_1}^3(\cos 3\gamma_1 + 3\cos\gamma_1) \\ &\quad + \tfrac{1}{4}{p_{12}}^3{A_2}^3(\cos 3\gamma_2 + 3\cos\gamma_2) \\ &\quad + \tfrac{3}{2}{p_{11}}^2 p_{12}{A_1}^2 A_2(\cos 2\gamma_1 + 1)\cos\gamma_2 \\ &\quad + \tfrac{3}{2}p_{11}{p_{12}}^2 A_1{A_2}^2\cos\gamma_1(\cos 2\gamma_2 + 1). \end{aligned} \tag{14}$$

In the postulated solution to Eq. (2), harmonic and cross-coupling terms have been neglected since only nonresonant modes are being considered. Hence, Eq. (14) is simplified to

$$\begin{aligned} {x_1}^3 &= \tfrac{3}{4}{p_{11}}^3{A_1}^3\cos\gamma_1 + \tfrac{3}{4}{p_{12}}^3{A_2}^3\cos\gamma_2 \\ &\quad + \tfrac{3}{2}{p_{11}}^2 p_{12}{A_1}^2 A_2\cos\gamma_2 + \tfrac{3}{2}p_{11}{p_{12}}^2 A_1{A_2}^2\cos\gamma_1. \end{aligned} \tag{15}$$

Namely,

$$\begin{aligned} {x_1}^3 &= \tfrac{3}{4}{p_{11}}^3{A_1}^2 y_1 + \tfrac{3}{4}{p_{12}}^3{A_2}^2 y_2 \\ &\quad + \tfrac{3}{2}{p_{11}}^2 p_{12}{A_1}^2 y_2 + \tfrac{3}{2}p_{11}{p_{12}}^2{A_2}^2 y_1. \end{aligned} \tag{16}$$

Similarly, ${x_2}^3$ may be written as

$$\begin{aligned} {x_2}^3 &= \tfrac{3}{4}{p_{21}}^3{A_1}^2 y_1 + \tfrac{3}{4}{p_{22}}^3{A_2}^2 y_2 \\ &\quad + \tfrac{3}{2}{p_{21}}^2 p_{22}{A_1}^2 y_2 + \tfrac{3}{2}p_{21}{p_{22}}^2{A_2}^2 y_1. \end{aligned} \tag{17}$$

The fifth-order term may be also simplified using an identical method. From Eq. (2),

$$\begin{aligned} {x_k}^5 &= \tfrac{5}{8}{p_{k1}}^5{A_1}^4 y_1 + \tfrac{5}{8}{p_{k2}}^5{A_2}^4 y_2 \\ &\quad + \tfrac{15}{8}{p_{k1}}^4 p_{k2}{A_1}^4 y_2 + \tfrac{15}{8}p_{k1}{p_{k2}}^4{A_2}^4 y_1 \\ &\quad + \tfrac{30}{8}{p_{k1}}^3{p_{k2}}^2{A_1}^2{A_2}^2 y_1 + \tfrac{30}{8}{p_{k1}}^2{p_{k2}}^3{A_1}^2{A_2}^2 y_2. \\ &\qquad (k = 1,2) \end{aligned} \tag{18}$$

And the seventh-order term may be also simplified using an identical method. From Eq. (2),

$$\begin{aligned} {x_k}^7 &= \tfrac{35}{64}{p_{k1}}^7{A_1}^6 y_1 + \tfrac{35}{64}{p_{k2}}^7{A_2}^6 y_2 \\ &\quad + \tfrac{35}{16}{p_{k1}}^6 p_{k2}{A_1}^6 y_2 + \tfrac{35}{16}p_{k1}{p_{k2}}^6{A_2}^6 y_1 \\ &\quad + \tfrac{105}{16}{p_{k1}}^5{p_{k2}}^2{A_1}^4{A_2}^2 y_1 + \tfrac{105}{16}{p_{k1}}^2{p_{k2}}^5{A_1}^2{A_2}^4 y_2 \\ &\quad + \tfrac{315}{32}{p_{k1}}^4{p_{k2}}^3{A_1}^4{A_2}^2 y_2 + \tfrac{315}{32}{p_{k1}}^3{p_{k2}}^4{A_1}^2{A_2}^4 y_1. \\ &\qquad (k = 1,2) \end{aligned} \tag{19}$$

Finally, the ninth-order term may be also simplified using an identical method. From Eq. (2),

$$\begin{aligned} x_k{}^9 &= \tfrac{63}{128}p_{k1}{}^9A_1{}^8y_1 + \tfrac{63}{128}p_{k2}{}^9A_2{}^8y_2 \\ &+ \tfrac{315}{128}p_{k1}{}^8p_{k2}A_1{}^8y_2 + \tfrac{315}{128}p_{k1}p_{k2}{}^8A_2{}^8y_1 \\ &+ \tfrac{315}{32}p_{k1}{}^2p_{k2}{}^7A_1{}^2A_2{}^6y_2 + \tfrac{315}{32}p_{k1}{}^7p_{k2}{}^2A_1{}^6A_2{}^2y_1 \\ &+ \tfrac{315}{16}p_{k1}{}^3p_{k2}{}^6A_1{}^2A_2{}^6y_1 + \tfrac{315}{16}p_{k1}{}^6p_{k2}{}^3A_1{}^6A_2{}^2y_2 \\ &+ \tfrac{945}{32}p_{k1}{}^4p_{k2}{}^5A_1{}^4A_2{}^4y_2 + \tfrac{945}{32}p_{k1}{}^5p_{k2}{}^4A_1{}^4A_2{}^4y_1. \\ &\qquad (k=1,2) \end{aligned} \tag{20}$$

Next, we define a square matrix of order 2 $\mathcal{Q}_a$, $\mathcal{Q}_b$, $\mathcal{Q}_c$ and $\mathcal{Q}_d$ as $\boldsymbol{x}_a = \mathcal{Q}_a\boldsymbol{y}$, $\boldsymbol{x}_b = \mathcal{Q}_b\boldsymbol{y}$, $\boldsymbol{x}_c = \mathcal{Q}_c\boldsymbol{y}$ and $\boldsymbol{x}_d = \mathcal{Q}_d\boldsymbol{y}$. Thus, using the relation of Eq. (5), Eq. (2) can be transformed to vector $\boldsymbol{y}$-form:

$$\begin{aligned} \frac{d^2\boldsymbol{y}}{d\tau^2} + (P^{-1}B_NP)\boldsymbol{y} &= -\varepsilon(\frac{d\boldsymbol{y}}{d\tau} - \frac{1}{3}\varepsilon_a(P^{-1}\mathcal{Q}_a)\frac{d\boldsymbol{y}}{d\tau} \\ &+\frac{1}{5}\varepsilon_b(P^{-1}\mathcal{Q}_b)\frac{d\boldsymbol{y}}{d\tau} - \frac{1}{7}\varepsilon_c(P^{-1}\mathcal{Q}_c)\frac{d\boldsymbol{y}}{d\tau} + \frac{1}{9}(P^{-1}\mathcal{Q}_d)\frac{d\boldsymbol{y}}{d\tau}) \end{aligned} \tag{21}$$

which may be rewritten, from Eq. (21), as

$$\begin{aligned} \frac{d^2\boldsymbol{y}}{d\tau^2} + (P^{-1}B_NP)\boldsymbol{y} &= -\varepsilon(\frac{d\boldsymbol{y}}{d\tau} - \frac{1}{3}\varepsilon_aR_a\frac{d\boldsymbol{y}}{d\tau} \\ &+ \frac{1}{5}\varepsilon_bR_b\frac{d\boldsymbol{y}}{d\tau} - \frac{1}{7}\varepsilon_cR_c\frac{d\boldsymbol{y}}{d\tau} + \frac{1}{9}R_d\frac{d\boldsymbol{y}}{d\tau}) \end{aligned} \tag{22}$$

where

$$P^{-1}\mathcal{Q}_k = R_k. \qquad (k=a,b,c,d) \tag{23}$$

The unperturbed equation (8) is completely separated by each mode. In perturbed equation, however, each mode is not separated completely as in Eq. (21). the main mode can be determined by

$$\begin{aligned} \frac{d^2y_i}{d\tau^2} + \lambda_iy_i &= -\varepsilon(\frac{dy_i}{d\tau} - \frac{1}{3}\varepsilon_a[R_a]_{ii}\frac{dy_i}{d\tau} \\ &+\frac{1}{5}\varepsilon_b[R_b]_{ii}\frac{dy_i}{d\tau} - \frac{1}{7}\varepsilon_c[R_c]_{ii}\frac{dy_i}{d\tau} + \frac{1}{9}[R_d]_{ii}\frac{dy_i}{d\tau}). \end{aligned} \tag{24}$$

As P is an orthogonal matrix and P^{-1} equals the transposed matrix (P^t), then we can consider

$$R_N = P^t\mathcal{Q}. \qquad (N=a,b,c,d) \tag{25}$$

Therefore, $[R_N]_i(\equiv [R_N]_{ii})$ are written as follows:

$$\begin{aligned} [R_a]_1 &= \tfrac{3}{8}A_1{}^2 + \tfrac{3}{4}A_2{}^2, \\ [R_a]_2 &= \tfrac{3}{8}A_2{}^2 + \tfrac{3}{4}A_1{}^2, \\ [R_b]_1 &= \tfrac{5}{32}A_1{}^4 + \tfrac{15}{32}A_2{}^2 + \tfrac{15}{16}A_1{}^2A_2{}^2, \\ [R_b]_2 &= \tfrac{5}{32}A_2{}^4 + \tfrac{15}{32}A_1{}^2 + \tfrac{15}{16}A_1{}^2A_2{}^2, \\ [R_c]_1 &= \tfrac{35}{512}A_1{}^6 + \tfrac{35}{128}A_2{}^6 + \tfrac{105}{128}A_1{}^4A_2{}^2 + \tfrac{315}{256}A_1{}^2A_4{}^4, \\ [R_c]_2 &= \tfrac{35}{512}A_2{}^6 + \tfrac{35}{128}A_1{}^6 + \tfrac{105}{128}A_1{}^2A_2{}^4 + \tfrac{315}{256}A_1{}^4A_4{}^2, \\ [R_d]_1 &= \tfrac{63}{2048}A_1{}^8 + \tfrac{315}{2048}A_2{}^8 + \tfrac{315}{512}A_1{}^6A_2{}^2, \\ &+ \tfrac{315}{256}A_1{}^2A_2{}^6 + \tfrac{945}{512}A_1{}^4A_2{}^4, \\ [R_d]_2 &= \tfrac{63}{2048}A_2{}^8 + \tfrac{315}{2048}A_1{}^8 + \tfrac{315}{512}A_1{}^2A_2{}^6 \\ &+ \tfrac{315}{256}A_1{}^6A_2{}^2 + \tfrac{945}{512}A_1{}^4A_2{}^4. \end{aligned} \tag{26}$$

Substituting Eq. (9) into Eq. (24) by assuming that the amplitude A_i and ϕ_i are slowly varying functions of time gives averaged equations:

$$\frac{dA_i{}^2}{d\tau} = -\varepsilon A_i{}^2(1-\frac{1}{3}\varepsilon_a[R_a]_i+\frac{1}{5}\varepsilon_b[R_b]_i+\frac{1}{7}\varepsilon_c[R_c]_i+\frac{1}{9}[R_d]_i). \tag{27}$$

3.3. Stability of the Double Modes

The mode stability is determined by introducing a small perturbation around the stationary point and determining whether all the eigenvalues of the resultant Jacobian:

$$J_{ij} = \frac{d}{dA_j{}^2}\frac{dA_i{}^2}{d\tau} \tag{28}$$

have negative real parts. The elements of Jacobian associated with Eq. (28) are

$$\begin{aligned} J_{11} &= -\varepsilon\{1 - \varepsilon_a(\tfrac{A_1{}^2}{4} + \tfrac{A_2{}^2}{4}) \\ &+\varepsilon_b(\tfrac{3}{32}A_1{}^4 + \tfrac{3}{32}A_2{}^4 + \tfrac{3}{8}A_1{}^2A_2{}^2) \\ &-\varepsilon_c(\tfrac{5}{128}A_1{}^6 + \tfrac{5}{128}A_2{}^6 + \tfrac{45}{128}A_1{}^4A_2{}^2 + \tfrac{45}{128}A_1{}^2A_2{}^4) \\ &+(\tfrac{35}{2048}A_1{}^8 + \tfrac{35}{2048}A_2{}^8 + \tfrac{35}{128}A_1{}^6A_2{}^2 \\ &+ \tfrac{35}{128}A_1{}^2A_2{}^6 + \tfrac{315}{512}A_1{}^4A_2{}^4)\}, \\ J_{12} &= -\varepsilon\{-\tfrac{\varepsilon_a}{4}A_1{}^2 + \varepsilon_b(\tfrac{3}{16}A_1{}^2A_2{}^2 + \tfrac{3}{16}A_1{}^4) \\ &-\varepsilon_c(\tfrac{15}{128}A_1{}^2A_1{}^4 + \tfrac{15}{128}A_1{}^6 + \tfrac{45}{128}A_1{}^4A_2{}^2) \\ &+(\tfrac{35}{512}A_1{}^2A_2{}^6 + \tfrac{35}{512}A_1{}^8 + \tfrac{105}{256}A_1{}^4A_2{}^4 + \tfrac{105}{256}A_1{}^6A_2{}^2)\}. \end{aligned} \tag{29}$$

J_{21} and J_{22} can be also obtained by changing $A_1{}^2$ to $A_2{}^2$, and p_{11} to p_{21}, and p_{12} to p_{22} in Eq. (29), respectively. For the double-mode to be stable, both A_1 and A_2 must exist. Then, from Eq. (9), Eq. (5) can be written

$$x_i = p_{i1}A_1\cos(\sqrt{\lambda_1}\tau+\phi_1)+p_{i2}A_2\cos(\sqrt{\lambda_2}\tau+\phi_2). \tag{30}$$

Setting Eq. (27) to zero, it can be shown that

$$\begin{aligned} &(A_1{}^2 - A_2{}^2)\{\tfrac{\varepsilon_a}{8} - \tfrac{\varepsilon_b}{16}(A_1{}^2 + A_2{}^2) \\ &+\tfrac{15\varepsilon_c}{512}(A_1{}^4 + 3A_1{}^2A_2{}^2 + A_2{}^4) \\ &-\tfrac{7}{512}(A_1{}^2 + A_2{}^2)(A_1{}^4 + 5A_1{}^2A_2{}^2 + A_2{}^4)\} = 0. \end{aligned} \tag{31}$$

Now, two cases are considered as follows:
1) The stationary value for amplitudes, when $A_1{}^2 = A_2{}^2$, are given by

$$1 - \tfrac{3}{8}\varepsilon_aA_1{}^2 + \tfrac{5}{16}\varepsilon_bA_1{}^4 - \tfrac{175}{512}\varepsilon_cA_1{}^6 + \tfrac{441}{1024}A_1{}^8 = 0. \tag{32}$$

When the parameter values are as $\alpha = 0.1$, $\varepsilon = 0.3$, $\varepsilon_a = 11.29$, $\varepsilon_b = 16.91$ and $\varepsilon_c = 7.53$,

$$A_1{}^2 \simeq 0.4, 1.3, 1.9, 2.4. \tag{33}$$

The corresponding stability matrix J for the double modes is written from Eq. (28) as

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{34}$$

where

$$J_{11} = J_{22} = -\varepsilon(1 - \tfrac{1}{2}\varepsilon_a {A_1}^2 + \tfrac{9}{16}\varepsilon_b {A_1}^4 - \tfrac{25}{32}\varepsilon_c {A_1}^6 + \tfrac{1225}{1024}{A_1}^8),$$
$$J_{12} = J_{21} = -\varepsilon(\tfrac{\varepsilon_a}{4}{A_1}^2 + \tfrac{6}{16}\varepsilon_b {A_1}^4 - \tfrac{75}{128}\varepsilon_c {A_1}^6 + \tfrac{490}{512}{A_1}^8). \quad (35)$$

The characteristic roots of Eq. (34) must be both negative for a stable double mode to exist. The conditions that have to be satisfied are

$$1 - \tfrac{3}{4}\varepsilon_a {A_1}^2 + \tfrac{15}{16}\varepsilon_b {A_1}^4 - \tfrac{175}{128}\varepsilon_c {A_1}^6 + \tfrac{2205}{1024}{A_1}^8 > 0 \quad (36)$$

and

$$1 - \tfrac{1}{4}\varepsilon_a {A_1}^2 + \tfrac{3}{16}\varepsilon_b {A_1}^4 - \tfrac{25}{128}\varepsilon_c {A_1}^6 + \tfrac{245}{1024}{A_1}^8 > 0. \quad (37)$$

Substituting Eq. (33) into Eq. (36) and Eq. (37), the cases of ${A_1}^2 \simeq 1.3$ and 2.4 are stable, however other cases of ${A_1}^2 \simeq 0.4$ and 1.9 are unstable.

$2)$ When
$\frac{\varepsilon_a}{8} - \frac{\varepsilon_b}{16}({A_1}^2 + {A_2}^2) + \frac{15\varepsilon_c}{512}({A_1}^4 + 3{A_1}^2{A_2}^2 + {A_2}^4) - \frac{7}{512}({A_1}^2 + {A_2}^2)({A_1}^4 + 5{A_1}^2{A_2}^2 + {A_2}^4\} = 0$,
and the parameter values are as $\alpha = 0.1$, $\varepsilon = 0.3$, $\varepsilon_a = 11.29$, $\varepsilon_b = 16.91$ and $\varepsilon_c = 7.53$, ${A_1}^2$ are given by

$$-0.13 + 2.2{A_2}^2 - 9.6{A_2}^4 + 16.7{A_2}^6 - 13.8{A_2}^8 + 5.2{A_2}^{10} - 0.4{A_2}^{12} - 0.3{A_2}^{14} + \tfrac{7}{128}{A_2}^{16} \simeq 0 \quad (38)$$

where

$${A_1}^2 \simeq 8.5 - 7.4{A_2}^2 + 2.0{A_2}^4. \quad (39)$$

From Eq. (38) are given by

$${A_2}^2 \simeq -4.6, 0.094, 0.32, 1.1, 1.9, 2.4. \quad (40)$$

From Eq. (39) are given by

$${A_1}^2 \simeq 85.6, 7.9, 6.4, 2.9, 1.7, 2.4. \quad (41)$$

The stability matrix J is

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \quad (42)$$

where

$$J_{11} = J_{22} = -\varepsilon(0.2 - 10.3{A_2}^2 + 42.6{A_2}^4 + 61.8{A_2}^6 - 39.9{A_2}^8 + 8.6{A_2}^{10} + 2.9{A_2}^{12} - 1.8{A_2}^{14} + \tfrac{35}{128}{A_2}^{16}),$$
$$J_{12} = -\varepsilon(21.1 - 121.8{A_2}^2 + 298.1{A_2}^4 + 414.3{A_2}^6 - 356.6{A_2}^8 + 195.0{A_2}^{10} + 66.3{A_2}^{12} - 12.9{A_2}^{14} + \tfrac{35}{32}{A_2}^{16}),$$
$$J_{21} = -\varepsilon(2.5{A_2}^2 + 12.1{A_2}^4 + 23.9{A_2}^6 - 25.0{A_2}^8 + 14.5{A_2}^{10} + 4.4{A_2}^{12} - \tfrac{35}{64}{A_2}^{14}). \quad (43)$$

In order to exist stable double modes, the conditions that have to be satisfied from Eq. (44)

$$J_{11} - \sqrt{J_{12}J_{21}} > 0. \quad (44)$$

(1)
(2)
(3)

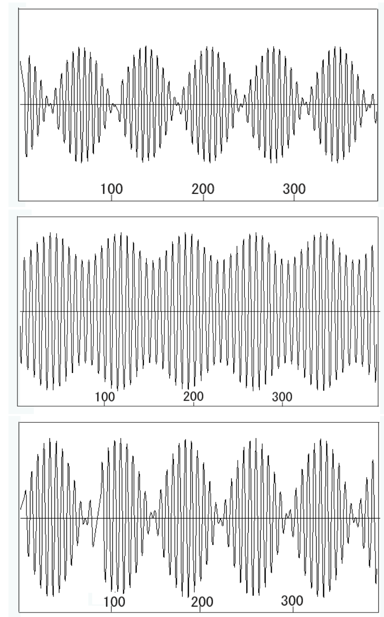


Fig. 3: Theoretical results of three double mode oscillation.

Substituting Eq. (40) into Eq. (44), the cases of ${A_2}^2 \simeq 0.32$ and 2.4 are stable, but other case of ${A_2}^2 \simeq 0.094$, 1.1 and 1.9 are unstable.

From $1)$ and $2)$, the theoretical results show that three different types of double mode oscillation coexist for a range of parameter values. Figure 3 shows theoretical results of three different types of double mode oscillation.

4. Conclusions

It has been shown that the Kryloff-Bogoliuboff linearization method applied to coupled oscillators can be used to analyze the stability of double mode oscillations. The theoretical results have shown that three different types of double mode coexist for a range of parameter values. These agree well with the results of computer simulations and circuit experiments.

References

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