

Finding Complex Conjugate Solutions of Nonlinear Algebraic Equations

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abstract: Numerical analysis of nonlinear algebraic equations is one of the most fundamental and important problems in scientific and engineering fields. In this paper, we propose an efficient SPICE-oriented algorithm for calculating the multiple solutions and bifurcation points. Firstly, a given nonlinear equation is replaced by the equivalent resistive circuit model using ABMs (analog behavior models) of SPICE. Therefore, the DC solutions correspond to those of the nonlinear equations. We apply the Newton homotopy method to solve the equations, where the homotopy paths are traced by an efficient algorithm based on the arc-length method. To find all the solutions, it is important to find the limit points and pitchfork bifurcation points, because the solution curve in the real space will be disappear at the limit points, and happens to appear the complex conjugate solutions from the point. Therefore, we propose an algorithm for calculating the solution curves in the complex space starts from the limit point. In such way, we can find out all the solutions in most of the cases.

1. Introduction

It is very important to calculate the multiple solutions of nonlinear equations, because there are many kind of resistive circuits such as flip-flops and Schmitt-trigger which have the multiple DC solutions. Unfortunately, a well-known Newton-Raphson method cannot any more be applied to find the multiple solutions. Therefore, there have been published many papers [1]-[5] for finding the multiple and/or all the solutions of the following nonlinear equation:

$$\left. \begin{aligned} f_1(v_1, v_2, \dots, v_n) &= 0 \\ f_2(v_1, v_2, \dots, v_n) &= 0 \\ &\vdots \\ f_n(v_1, v_2, \dots, v_n) &= 0 \end{aligned} \right\} \quad (1)$$

Some of them [1]-[2] can efficiently find all the solutions of piecewise-linear resistive circuits, where the nonlinear elements are modeled by the piecewise-linear segments. Although the interval methods [3]-[4] can also find all the solutions, it is rather time-consuming for large scale systems. The homotopy methods [5] are usefully applied to calculate the multiple solutions of nonlinear equations.

There have been proposed many efficient algorithms using SPICE simulators [6]-[8] whose algorithms are based on the homotopy methods. Although they can efficiently find the multiple solutions, it is not known whether they have gotten all the solutions or not. The homotopy paths may consist of the multiple independent branches and the closed loops, whose branches may cross at the points called pitchfork bifurcation points. Unfortunately, the curve tracing algorithm based on arc-length method [10] may fail to trace the curve, because the rank of the Jacobian matrix of eq.(1) is reduced to equal or less than $n - 1$ at the points. On the other hand, since the rank is equal to n at the limit point, we can successfully continue to trace the homotopy path. The bifurcation points belong to the singular points, so that the curve may turn or cross the other curves there. Therefore, it is important to calculate the bifurcation points to get multiple solutions or to know the properties of nonlinear equation (1).

In this paper, we show the curve tracing algorithm using SPICE in section 2, and SPICE-oriented algorithms to find the limit and pitchfork bifurcation points in section 3. We also show that the limit point in the real space also corresponds to the limit point of the complex space. We found an interesting property such that, by tracing the solution curve in the complex space, we can find another isolated closed loop in the real space. In this way, we can calculate the multiple solutions or all the solution of nonlinear algebraic equations. The interesting illustrative examples are shown section 3 and 4.

2. Curve tracing algorithm

Nowadays, SPICE is widely used circuit simulations of integrated circuits. Since nonlinear devices in ICs are modeled by many kinds of the functions, SPICE has all kinds of the controlled sources, and nonlinear functions such as exponential functions, multiplications, additions and subtractions and so on. Furthermore, we can easily get any function using ABMs(analog behav-

ior models) in such manner that the desired functions are programmed with Fortran language. An example of single-output function having the multi-inputs is shown in Fig.1, where the function $f(v_1, v_2, \dots, v_n)$ in the black box is realized by writing the function with the input variables $\{v_1, v_2, \dots, v_n\}$. On the other hand, a set of the nonlinear equations given by (1) is realized by the equivalent circuit as shown in Fig.2.

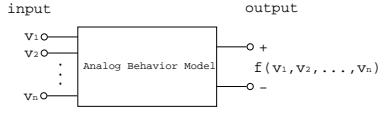


Figure 1: Analog Behavior Model

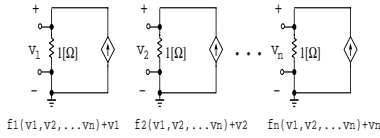


Figure 2: Equivalence circuit of the nonlinear algebraic equation

Namely, if we first assume the voltage at i th resistor (1Ω) voltage v_i , we can realize the current source.

$$I_i = f_i(v_1, v_2, \dots, v_n) - v_i, \quad i = 1, 2, \dots, n \quad (2)$$

Thus, we have $f_i(v_1, v_2, \dots, v_n) = 0$ with the Kirchoff's voltage law. Let us apply the *Newton homotopy method* to solve the equation (1). Then, we formulate the following modified relation:

$$\mathbf{F}(\mathbf{v}, \rho) = \mathbf{f}(\mathbf{v}) + (\rho - 1)\mathbf{I}_0 = \mathbf{0} \quad (3)$$

where $\mathbf{f}(\mathbf{v}) : \mathbf{R}^n \mapsto \mathbf{R}^n$ given by the relation (1), and $\mathbf{I}_0 = \mathbf{f}(\mathbf{v}_0)$ is the initial guess. Thus, the relation (2) is consisted “ n ” equations having “ $n+1$ ” variables, so that the solution satisfying the relation (3) is given by the solution curves in “ $(n+1)$ ” dimensional space. In order to trace the solution curve with the arc-length method, we apply the following relation [9]:

$$\sum_{i=1}^K \left(\frac{dv_i}{ds} \right)^2 = 1 \quad (4)$$

Observe that the solutions satisfying (1) are obtained at $\rho = 1$ on the solution curve of eq.(3). Now, we show the equivalent circuit satisfying the Newton homotopy method (3) and (4). The block diagram is shown by Fig.3(a). Some of the “ K ” node voltages in the equation (1) are chosen as the variables of the arc-length method (4) [8]. Thus, the circuit realizing the arc-length method is shown by Fig.3(b). that is called STC(solution curve tracing circuit)[6]. In many practical problems [8], we can get all the DC solutions from a suitable initial guess \mathbf{I}_0 in equation (3).

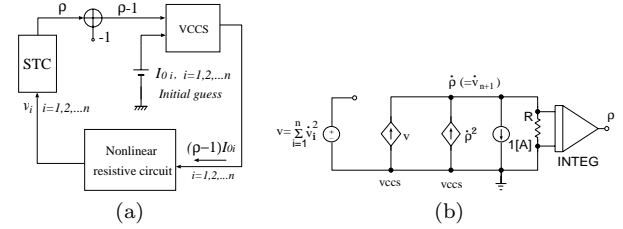


Figure 3: (a)Equivalence circuit of Newton homotopy method. (b)Solution curve tracing circuit.

3. Calculations of bifurcation points

Now, consider an algorithm for calculating the bifurcation points of the nonlinear equations. In this section, we show an efficient SPICE-oriented algorithm. For simplicity, we set $v_{n+1} = \rho$ and rewrite the relation (3) as follows:

$$\mathbf{F}(\mathbf{v}) = \mathbf{0} \quad \mathbf{F} : \mathbf{R}^{n+1} \mapsto \mathbf{R}^n \quad (5)$$

It is proved that our algorithm [9] can trace the solution curves only if the rank of the Jacobian matrix of (5) is n on the whole curve. Since the rank at the limit points is equal to “ n ”, and less than or equal to $n - 1$ at the pitchfork bifurcation points. Therefore, the limit points can be traced by our arc-length method. However, it is usually not so easy to calculate the Jacobian matrix and the rank. Thus, let us to obtain an algorithm to calculate the bifurcation points without estimating the Jacobian matrix.

For the pitchfork point of (1), we consider a set of the following modified equations¹ :

$$\left. \begin{aligned} \mathbf{F}_1(\mathbf{v}, h) &= \mathbf{F}(\mathbf{v}) + h\mathbf{I} = \mathbf{0} \\ \mathbf{F}_2(\mathbf{v}, \Delta\mathbf{v}) &= \mathbf{F}(\mathbf{v} + \Delta\mathbf{v}) - \mathbf{F}(\mathbf{v}) = \mathbf{0} \end{aligned} \right\} \quad (6)$$

which have $2n+2$ variables and $2n$ equations if we assume $\Delta\mathbf{v}$ as the additional variables. Since the rank of the Jacobian matrix of $\mathbf{F}(\mathbf{v})$ is equal or less than $n - 1$ [10], if we set the variables Δv_1 and Δv_2 small fixed constants in $\mathbf{F}_2(\mathbf{v}, \Delta\mathbf{v})$, then, eq.(6) is the nonlinear equation with “ $2n$ ” variables and “ $2n$ ” equations. Hence, we also apply the Newton homotopy method as follows:

$$\left. \begin{aligned} \mathbf{F}_1(\mathbf{v}, h) + (\lambda - 1)\mathbf{I}_{10} &= \mathbf{0} \\ \mathbf{F}_2(\mathbf{v}, \Delta\mathbf{v}) + (\lambda - 1)\mathbf{I}_{20} &= \mathbf{0} \\ \sum_{i=1}^K \left(\frac{dv_i}{ds} \right)^2 + \sum_{i=3}^K \left(\frac{d\Delta v_i}{ds} \right)^2 + \left(\frac{dh}{ds} \right)^2 + \left(\frac{d\lambda}{ds} \right)^2 &= 1 \end{aligned} \right\} \quad (7)$$

Tracing the homotopy path (7), we can find the bifurcation points at $\lambda = 1$ and $h = 0$. Note that two of the variables Δv_1 and Δv_2 and the initial guess \mathbf{I}_{20} must be chosen the fixed sufficient small values.

¹ At the pitchfork points, the second order term of the Taylor expansions are assumed negligible values for the small variations

Example 3.1. Now, let us calculate bifurcation points of the following equation:

$$f(v_1, v_2) = (v_1^3 - 4v_1 - v_2)(-\frac{1}{4}v_1^3 - v_2 + 1) = 0 \quad (8)$$

Thus, we have the following modified relations:

$$\left. \begin{aligned} f_1(v_1, v_2, h) &= f(v_1, v_2) + h = 0 \\ f_2(v_1, v_2) &= f(v_1 + \Delta v_1, v_2) - f(v_1, v_2) = 0 \\ f_3(v_1, v_2) &= f(v_1, v_2 + \Delta v_2) - f(v_1, v_2) = 0 \end{aligned} \right\} \quad (9)$$

Note that (8) consists of two variables and one equation, and that it satisfies both $\frac{\partial f(v_1, v_2)}{\partial v_1} = 0$ and $\frac{\partial f(v_1, v_2)}{\partial v_2} = 0$ at the pitchfork points. Hence, the fixed small variables Δv_1 and Δv_2 can be set as shown in (9). Applying the Newton homotopy method, we have

$$\left. \begin{aligned} f_1(v_1, v_2, h) + (\lambda - 1)I_{01} &= 0 \\ f_2(v_1, v_2) + (\lambda - 1)I_{02} &= 0 \\ f_3(v_1, v_2) + (\lambda - 1)I_{03} &= 0 \\ \left(\frac{dv_1}{ds}\right)^2 + \left(\frac{dv_2}{ds}\right)^2 + \left(\frac{dh}{ds}\right)^2 + \left(\frac{d\lambda}{ds}\right)^2 &= 1 \end{aligned} \right\} \quad (10)$$

where we chose $\Delta v_1 = \Delta v_2 = 10^{-5}$. Then, we have 3 pitchfork bifurcation points (2.0, 0.0), (-2.0, 0.0) and (-0.25, 0.98) at $h = 0$, $\lambda = 1$ as shown in Fig.4.

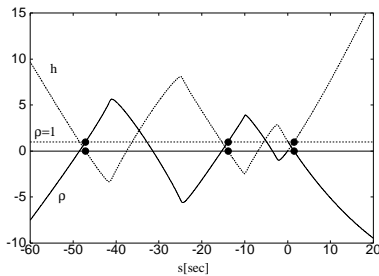


Figure 4: Solution of the pitch fork point

The algorithm is much more simple compared to the method [11] using the Jacobian matrix of (5), especially for the large scale systems.

Now, consider to calculate the branches starting from the pitchfork bifurcation points in (5). We set a small sphere centered at the bifurcation points \hat{v} , and get the intersections between the branches of (5) and the sphere. Thus, we have

$$\mathbf{F}(\mathbf{v}) = \mathbf{0} \quad , \quad \left. \sum_{i=1}^K (v_i - \hat{v}_i)^2 = \gamma \right\} \quad (11)$$

for a sufficient small positive value γ . The relations (11) are also solved by the Newton homotopy method. The results for the above example are shown by Fig.5(a),(b) and (c), respectively. Starting from these intersections, we can trace the solution curves of (5) with arc-length method [9].

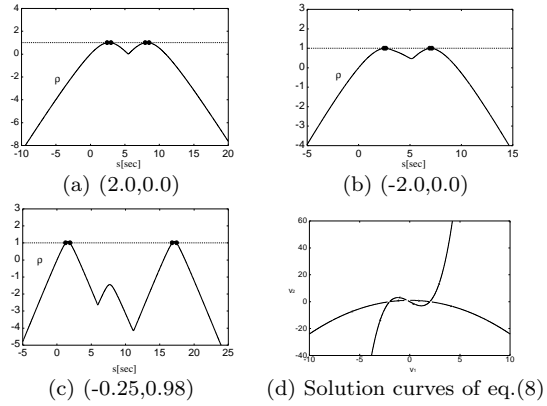


Figure 5:

4. Solution curves of algebraic equations in complex space

We consider the solution curves of the following equation:

$$f(v_1, v_2) = \sum_{i=0}^N \sum_{j=0}^M a_{ij} v_1^i v_2^j = 0 \quad (12)$$

Assume that the solution curves may consist of several *disjoint loops* and/or *independent branches*. The independent branches with a pitchfork point can be found by the above method in section 3. In this section, we will develop an algorithm to find the disjoint loops. Equation (12) satisfies the following relations:

$$\frac{\partial v_2}{\partial v_1} = 0 \quad \text{or equivalently} \quad \frac{\partial f(v_1, v_2)}{\partial v_1} = 0 \quad (13)$$

at the limit points. Therefore, the points can be also found by solving

$$\left. \begin{aligned} f_1(v_1, v_2, h) &= f(v_1, v_2) + h = 0 \\ f_2(v_1, v_2) &= f(v_1 + \Delta v_1, v_2) - f(v_1, v_2) = 0 \end{aligned} \right\} \quad (14)$$

by choosing a sufficiently small fixed constant Δv_1 . Thus, the limit point (\hat{v}_1, \hat{v}_2) can be obtained by solving (14). Note that when the value v_2 is increased or decreased from the limit point \hat{v}_2 , the solutions v_1 of (12) will be given by the complex conjugate values or the real values. We consider here the solution curves in the complex space. In this case, we set the variables $(a + jb, v_2)$ instead of (v_1, v_2) in (12). The directions of the branches are found by the intersections of a small sphere with the branches of (12) as follows:

$$\left. \begin{aligned} \Re\{f((a + jb), v_2)\} &= 0 \\ \Im\{f((a + jb), v_2)\} &= 0 \\ (\hat{v}_1 - a)^2 + b^2 + (v_2 - \hat{v}_2)^2 &= r^2 \end{aligned} \right\} \quad (15)$$

for a small "r". Thus, the solution curves in the complex space can be found by solving the first two equations in (15). Remark that a new solution curve satisfying (12) in real space will be also appeared at another limit point in the complex space. Thus, the other disjoint loop of (15) in real space can be found in the same way.

Example 4.1 Now, consider a simple example as follows:

$$F(v_1, v_2) = v_1^4 - v_1^3 - 2v_1^2 - 2v_1 - v_2 + 4 = 0 \quad (16)$$

For finding the limit point, we calculate the Jacobian

$$\frac{\partial F(v_1, v_2)}{\partial v_1} = 4v_1^3 - 3v_1^2 - 4v_1 - 2 = 0 \quad (17)$$

The real solution of (17) corresponding to the limit point of (16) is given by $(\hat{v}_1 = 1.58, \hat{v}_2 = -1.81)$. Now, set $v_1 = a + jb$ in (16) for getting the solution curve in complex space. Then, we have

$$\left. \begin{aligned} \Re\{F(a + jb, v_2)\} &= a^4 - a^3 - 2a^2 - 2a + 4 \\ &\quad - (6a^2 - 3a - 2)b^2 + b^4 - v_2 = 0 \\ \Im\{F(a + jb, v_2)\} &= 4a^3 - 3a^2 - 4a - 2 - (4a - 1)b^2 = 0 \end{aligned} \right\} \quad (18)$$

We assume a small sphere centered at the limit point as follows:

$$(a - \hat{v}_1)^2 + b^2 + (v_2 - \hat{v}_2)^2 = r^2 \quad (19)$$

Substituting $(\hat{v}_1 = 1.58, \hat{v}_2 = -1.81)$ and $r = 0.1$ into (19) and solving (18) together with (19), we have the intersection points $(a = 1.58, b = \pm 0.0245, v_2 = -1.87)$. Starting from the intersections on the sphere, we can get the solution curve in the complex space as shown in Fig.6.

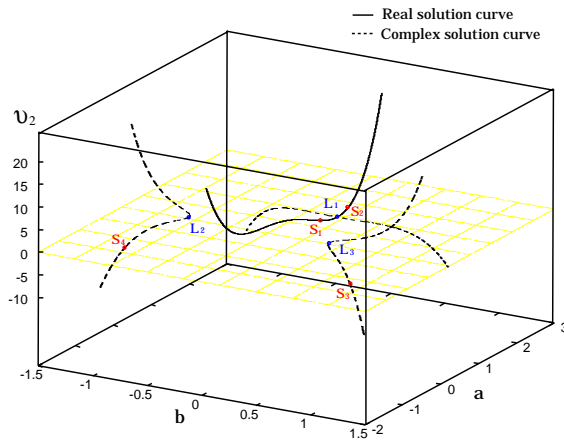


Figure 6: Solution curves in the complex space

5. Conclusions and remarks

It is very important to find all the solutions of nonlinear equations. In this paper, we have shown Newton homotopy method for finding the multiple solutions of nonlinear equations. The solutions are located on the solution curves satisfying the modified equation. The branches may consist of the disjoint and/or closed loops. We proposed an efficient SPICE-oriented algorithm for calculating the bifurcation points such as limit and/or pitchfork points. After then, the intersections of a small sphere centered at the bifurcation points with solution

curves are calculated. Thus, the solution curves starting from these points are obtained by the arc-length method [9]. The homotopy paths of the nonlinear equation sometimes consist of the disjoint closed loops. To find the multiple loops, we need to trace the solution curves in complex space starting from the limit points. We also showed an interesting result that the limit point in real space is also the limit point on the solution curves in the complex space. Although it is impossible to prove that all the solutions of the nonlinear algebraic equations are found with our method, all solutions can be found in many practical problems.

In the future problems, we need to extend the algorithm to large dimensional cases, and also extend to the nonlinear equation instead of the nonlinear algebraic equations.

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