Generation Mechanisms of Two Pairs of Waves in a Ladder of Oscillators

Masayuki YAMAUCHI, Yoshifumi NISHIO and Akio USHIDA

Dept. of Electrical and Electronic Engineering, Tokushima University, Japan
e-mail {masa,nishio,ushida}@ee.tokushima-u.ac.jp

1. Introduction

Large number of coupled limit-cycle oscillators is useful as models for a wide variety of systems in natural fields, for example, diverse physiological organs including gastrointestinal tracts and axial fiber of nervous systems, convecting fluids, arrays of Josephson junctions and so on. Hence, it is very important to analyze synchronization and the related phenomena observed in coupled oscillators in order to clarify mechanisms of the generations or in order to control the generating-conditions of various phenomena in such natural systems [1]. In the field of the electrical engineering, a lot of studies on synchronization phenomena of coupled van der Pol oscillators have been carried out up to now [2],[3].

Recently, we have discovered very interesting wave propagation phenomena of phase states between two adjacent oscillators in an array of van der Pol oscillators coupled by inductors [4]-[6]. In the study, we named the continuously existing wave of changing phase states between two adjacent oscillators from in-phase to anti-phase or from anti-phase to in-phase as "phase-inversion-wave" and explained the generation mechanism by using the relation between the instantaneous oscillation frequencies and the phase differences of two adjacent oscillators.

In this study, a propagating wave of phase difference between two adjacent oscillators is observed. It is called "phase-wave." The phase-waves are observed in transient states and change to the phase-inversion-waves or disappear. In the computer calculations, we produce phase-waves as follows. Almost same initial conditions are given for all oscillators to produce complete in-phase synchronization which is one of stable steady states in the system. After the system settles in the complete in-phase synchronization, the voltage and the current with arbitrary phase difference are input to the first oscillator. The observed phase-waves in coupled 20 oscillators are classified to four patterns and the difference is analyzed in detail. We observe that the behavior of the phase-waves generated by giving phase difference of plus value is different from those generated by giving phase difference of minus value. We can also observe the generation of two pairs of the phase-waves. The mechanisms of the complicated phenomena on the phase-waves are also explained.

2. Circuit Model

The circuit model used in this study is shown in Fig. 1. N van der Pol oscillators are coupled by coupling inductors

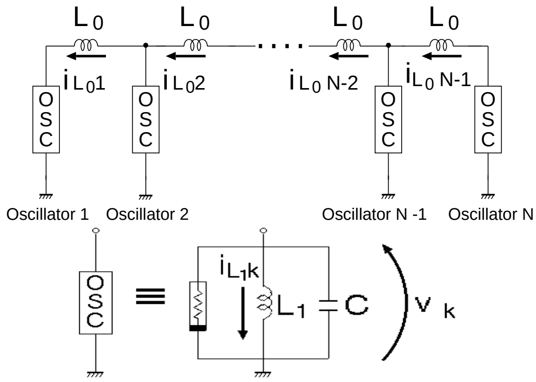


Figure 1: Coupled van der Pol oscillators as a ladder.

L_0. We carried out computer calculations for the cases of $N = 20$. In the computer calculations, we assume the $v - i$ characteristics of the nonlinear negative resistors in each circuit as the following function.

$$i_r(v_k) = -g_1 v_k + g_3 v_k^3 \quad (g_1, g_3 > 0). \tag{1}$$

The circuit equations governing the circuit in Fig. 1 are written as:

[First Oscillator]

$$\dot{x}_1 = y_1 \tag{2}$$

$$\dot{y}_1 = -x_1 + \alpha(x_2 - x_1) + \varepsilon\left(y_1 - \frac{1}{3}y_1^3\right)$$

[Middle Oscillators]

$$\dot{x}_k = y_k \tag{3}$$

$$\dot{y}_k = -x_k + \alpha(x_{k+1} - 2x_k + x_{k-1}) + \varepsilon\left(y_k - \frac{1}{3}y_k^3\right)$$

$(k = 2 \sim N\text{–}1)$

[Last Oscillator]

$$\dot{x}_N = y_N \tag{4}$$

$$\dot{y}_N = -x_N + \alpha(x_{N-1} - x_N) + \varepsilon\left(y_N - \frac{1}{3}y_N^3\right)$$

where

$$t = \sqrt{L_1 C}\tau \ , \ i_{L_1 k} = \sqrt{\frac{Cg_1}{3L_1 g_3}} x_k \ , v_k = \sqrt{\frac{g_1}{3g_3}} y_k \ ,$$

$$\alpha = \frac{L_1}{L_0} \ , \ \varepsilon = g_1 \sqrt{\frac{L_1}{C}} \ , \ \frac{d}{d\tau} = \text{“ . ”} \ . \quad (5)$$

It should be noted that α corresponds to the coupling of the oscillators and ε corresponds to the nonlinearity of the oscillators. Throughout the paper, we fix $\alpha = 0.050$ and $\varepsilon = 0.30$ and calculate (2)-(4) by using the fourth-order Runge-Kutta method.

3. Phase Wave

3.1. Classification

Observed phenomena are classified into four patterns by the input phase difference as follows:

(a) **No reflection**
$-67^\circ \le \Phi_{in} \le +64^\circ$: The phase difference decreases while the phase-waves are propagating. The phase-waves disappear at the edge of the array (see Fig. 2(a)).

(b) **One time reflection**
$+65^\circ \le \Phi_{in} \le +70^\circ$: The phase-waves propagate and reflect at the edge of the array. The phase difference decreases while the phase-waves are propagating after the reflection. The phase-waves disappear at the edge of the array (see Fig. 2(b)).

(c) **Two times reflection**
$\Phi_{in} \cong +71^\circ$: The phase-waves propagate and reflect two times at the edges of the array. The phase difference decreases while the phase-waves are propagating. The phase-waves disappear at the edge of the array (see Fig. 2(c)).

(d) **Change to phase-inversion-waves**
$+72^\circ \le \Phi_{in} \le +180^\circ$ and $-68^\circ \le \Phi_{in} \le -180^\circ$: The phase difference increases while the phase-waves are propagating. The phase-waves change to the phase-inversion-waves when the phase-waves reflect at the edge of the array or in the middle of the array (see Fig. 2(d)).

Figure 3 shows domain of the observed phenomena. We may be able to observe a phenomenon that the phase-waves reflect three times or more and disappear by changing parameter. When N is very large, we can observe a phenomenon that the phase-waves disappear in the middle of the array.

3.2. Two Pairs of Phase-Waves

Two pairs of waves (two pairs of phase-waves or a pair of phase-waves and a pair of phase-inversion-waves) are generated in the domain marked with diagonal line of Fig. 3 (see Fig. 4). For $\Phi_{in} = -68^\circ$, two pairs of phase-waves are generated as shown in Fig. 4(a). For $\Phi_{in} = -70^\circ$, a pair of phase-waves and a pair of phase-inversion-waves are generated as Fig. 4(b). For $\Phi_{in} = -69^\circ$, two pairs of phase-inversion-waves exist as Fig. 4(c). But, when waves are generated, a

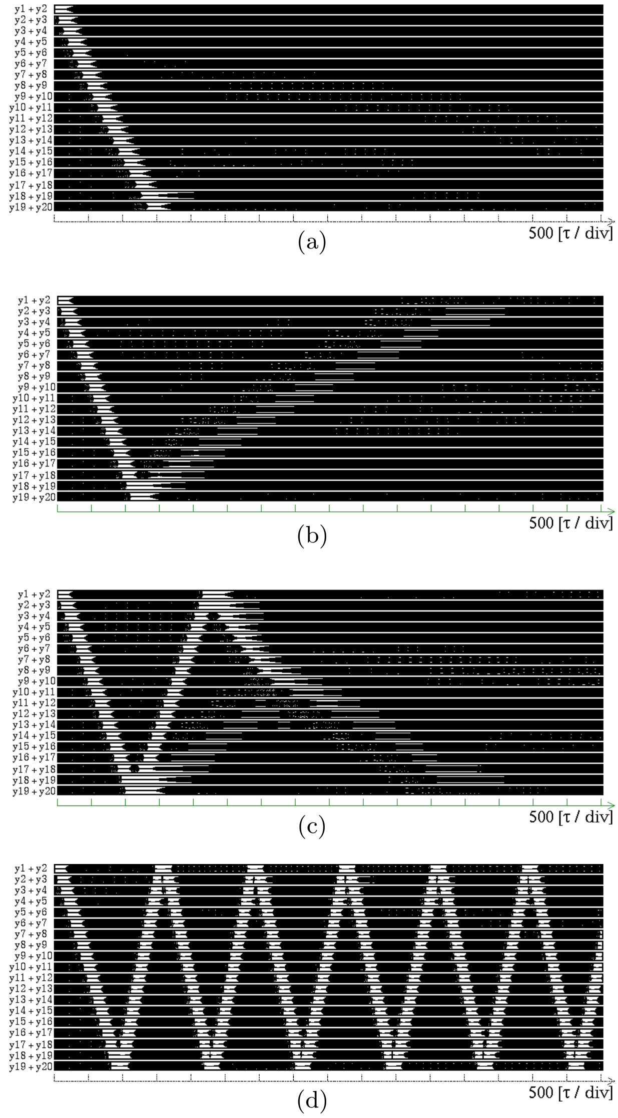


Figure 2: Phase-waves. $\alpha = 0.050$ and $\varepsilon = 0.30$. (a) $\Phi_{in} = +60^\circ$. (b) $\Phi_{in} = +68^\circ$. (c) $\Phi_{in} = +71^\circ$. (d) $\Phi_{in} = +80^\circ$.

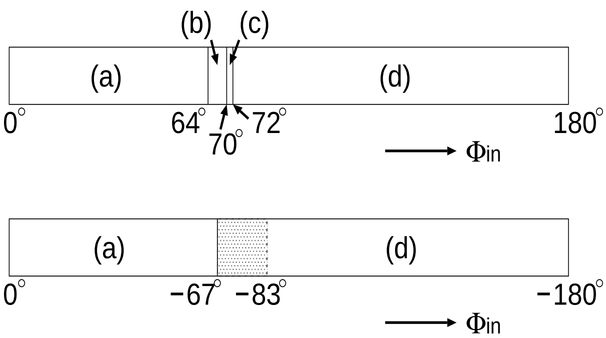
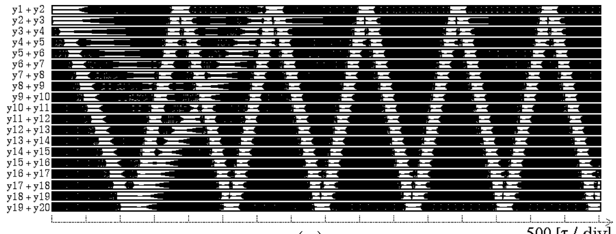
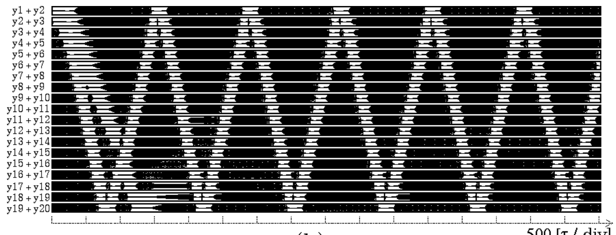


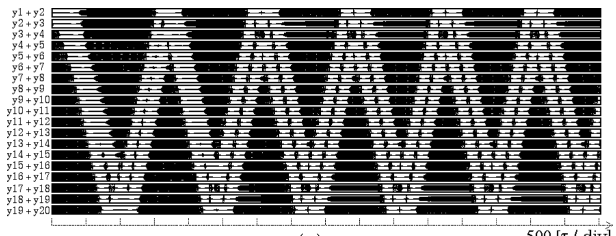
Figure 3: Domain of observed phenomena generated by the phase difference of plus or minus value. $\alpha = 0.050$ and $\varepsilon = 0.30$.



(a)

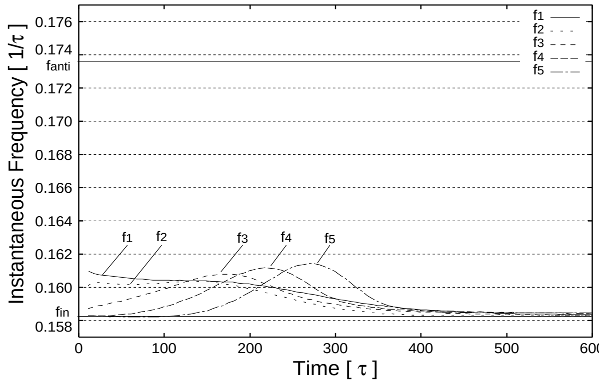


(b)

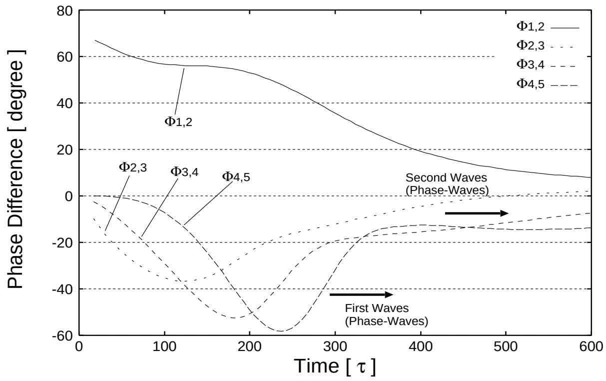


(c)

Figure 4: Two pairs of phase-waves. $\alpha = 0.050$ and $\varepsilon = 0.30$. (a) $\Phi_{in} = -68°$. (b) $\Phi_{in} = -70°$. (c) $\Phi_{in} = -69°$.



(a) Frequency.



(b) Phase-Difference.

Figure 5: Mechanism of simultaneously generated two pairs of the phase-waves. Example of generated phase-waves by phase difference of minus value ($\Phi_{in} = -68°$).

pair of phase-inversion-waves are phase-waves and they change to phase-inversion-waves. Therefore, Fig. 4(c) has the same mechanism as Fig. 4(b) when waves are generated.

Further, two pairs of waves can be generated when waves reflect at the first time. This phenomenon is observed when phase difference at the reflection exists in the domain marked with diagonal line in Fig. 3. Therefore, the mechanism of this phenomenon is the same as the mechanism of generating two pairs of waves.

3.3. Generation of Two Pairs of Phase-Waves

We can observe two types of two pairs of phase-waves (see section 3.3);

1. Two pairs of phase-waves (Fig. 4(a))
2. A pair of phase-waves and a pair of phase-inversion-waves (Figs. 4(b) and (c)).

3.3.1. Mechanism of Figure 5 (a)

When all oscillators are in in-phase synchronization, phase difference of minus is given to the first oscillator.

(1) Because $\Phi_{1,2}$ is not equal to 0, f_1 and f_2 is larger than f_{in}. But, f_3 is equal to f_{in}, because $\Phi_{2,3}$ and $\Phi_{3,4}$ are equal to 0.

(2) $\Phi_{1,2}$ is attracted to 0 in a moment, because $\Phi_{1,2}$ is not large enough to be attracted to anti-phase synchronization. Therefore, f_1 and f_2 change toward f_{in}.

(3) $\Phi_{2,3}$ changes from 0, because f_2 is higher than f_{in}. Therefore, f_3 starts to change toward f_{anti}.

<first phase-waves>

(4) Propagating of the first phase-waves begin.

<secound phase-waves>

(4) $\Phi_{2,3}$ is decreasing during $f_2 > f_3$.

(5) f_2 starts to change from decreasing to f_{in} to increasing to f_{anti} again.

(6) f_1 changes toward f_{in} and becomes nearly equal to f_2, because transition of f_2 is very slow by the above reason. Therefore, $\Phi_{1,2}$ keeps almost same value during that the first pair of phase-waves propagate.

(7) f_2 starts to change to increase to f_{in} again. f_1 and f_2 decrease to f_{in} together, because $\Phi_{1,2}$ is not large enough to be attracted to anti-phase synchronization. Therefore, f_1, f_2 and $\Phi_{1,2}$ decrease slowly. The second pair of phase-waves generates at this time.

(8) $\Phi_{2,3}$ changes toward 0, because $f_2 < f_3$.

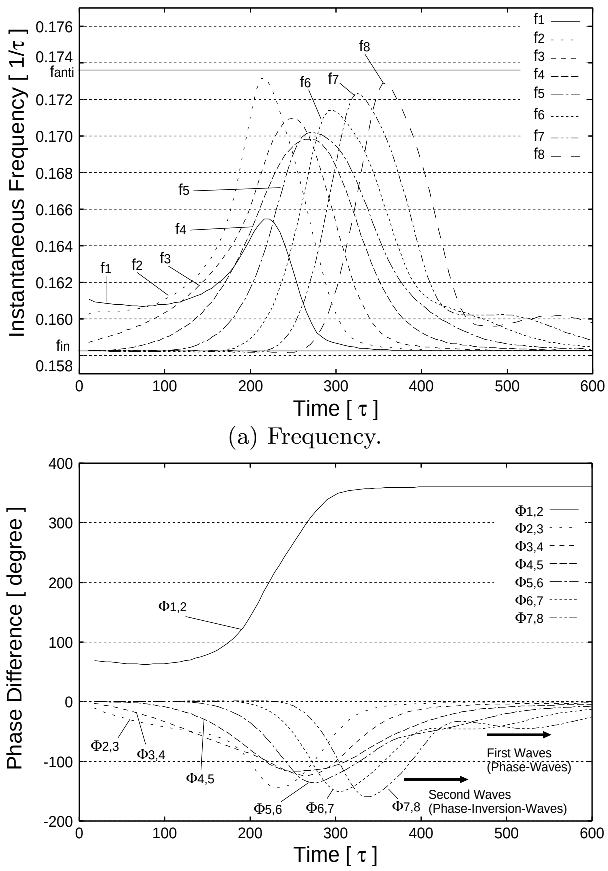
(9) f_3 starts to change toward f_{in}.

(10) Rate of decrease of $\Phi_{2,3}$ changes to small value, because f_3 becomes nearly equal to f_2 which decrease slowly. The second pair of phase-waves propagate by the above mechanism.

Two pairs of the phase-waves generate by this mechanism.

Figure 7 shows how the instantaneous frequencies and the phase differences change when two pairs of the phase-waves generate.

3.3.2. Mechanism of Figures 5 (b) and (c)



(a) Frequency.

(b) Phase-Difference.

Figure 6: Mechanism of simultaneously generated a pair of the phase-waves and a pair of the phase-inversion-waves. Example of generated phase-waves by phase difference of minus value ($\Phi_{in} = -70°$).

Mechanism of the first phase-waves is the same as Fig. 4(a). But, rate of increase of f_2 is larger than that when two pairs of phase-waves generate, because minimum value of $\Phi_{2,3}$ by the first pair of phase-waves is larger than that when two pairs of phase-waves generate (around $\tau = 100$ in Figs. 5(b) and 6(b)). Mechanism of the second waves is as follows:

(1) $\Phi_{1,2}$ changes from decrease to increase, because f_2 begins to increase quickly by $\Phi_{2,3}$ which decreases.

(2) f_1 changes toward f_{anti}.

(3) f_1 increases to middle of f_{in} and f_{anti} and decrease to f_{in} again, because edge of the array does not become stable in anti-phase synchronization. $\Phi_{1,2}$ exceeds 180° and reaches 360° (in-phase synchronization).

(4) The second pair of phase-waves is same waves which are generated when $\Phi_{in} = 180°$. Therefore, a pair of phase-waves changes to a pair of phase-inversion-waves in a moment. This mechanism is same mechanism of reflection of the phase-inversion-wave [4].

Figure 8 shows how the instantaneous frequencies and the phase differences changes when a pair of the phase-waves and a pair of phase-inversion waves generate.

4. Conclusions

In this study, the phase-waves were investigated by numerical calculation for the case of coupled 20 oscillators as a ladder. The observed phase-waves were classified to 4 patterns and the difference was analyzed in detail. We observed that the behavior of the phase-waves generated by giving phase difference of plus value was different from those generated by giving phase difference of minus value. We were able to observe the generation of two pairs of the phase-waves. Further, the mechanism of the complicated phenomena on the phase-waves were explained.

References

[1] L. L. Bonilla, C. J. Pérez Vicente and R. Spigler, "Time-Periodic Phases in Populations of Nonlinearly Coupled Oscillators with Bimodal Frequency Distributions" Physica D: Nonlinear Phenomena, vol. 113, issues 1, pp. 79–97, Feb. 1998.

[2] T. Suezaki and S. Mori, "Mutual Synchronization of Two Oscillators," Trans of IECE, vol. 48, no. 9, pp. 1551–1557, Sep. 1965.

[3] T. Endo and S. Mori, "Mode Analysis of a Multimode Ladder Oscillator," IEEE Trans. Circuits and Syst., vol. 23, pp. 100-113, Feb. 1976.

[4] M. Yamauchi, M. Wada, Y. Nishio and A. Ushida, "Wave Propagation Phenomena of Phase States in Oscillators Coupled by Inductors as a Ladder," *IEICE Trans. Fundamentals*, vol.E82-A, no.11, pp.2592-2598, Nov. 1999.

[5] M. Yamauchi, M. Wada, Y.Nishio and A. Ushida, "Collisions between Two Phase-Inversion-Waves in an Array of Oscillators," *Proc. ISCAS'00*, vol.1, pp.679-682, May 2000.

[6] M. Yamauchi, Y. Nishio and A. Ushida, "Reflection and Transmission of Phase-Inversion-Waves in Oscillators Coupled by Two Kinds Inductors," *Proc. ISCAS'01*, vol.3, pp.767-770, May 2001.