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Analysis of Oscillator Circuits Having Multiple Oscillations

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1. Introduction

Analysis of oscillator circuits is very important for designing communication circuits such as modulators and mixers, where one of the signals is coming from the internal oscillator and the other from the input signal. Therefore, we need to know the exact oscillator frequency in order to calculate the intermodulation. There are many type oscillators such as Colpitts, crystal oscillator and so on, each of which has a unique oscillator frequency. On the other hand, the coupled oscillators with the same oscillator frequency may have complicated phenomena depending on the initial conditions, some of which have many different phases and the same oscillating frequency [1]-[2]. These phenomena can be analytically explained in the case of the weakly nonlinear coupled oscillators [1]. Furthermore, it is known that the coupled oscillators may have quasi-periodic oscillations and/or chaos phenomena [3]. In this paper, we consider the steady-state analysis of oscillators with multiple mode oscillations. There have been published many papers discussing the steady-state analysis of oscillator circuits [4]-[8]. The references [5]-[7] are based on the time-domain techniques, where the initial guess giving rise to the steady-state response is found by the Newton-like shooting method. Especially, W.Ma, L.Trajkovic and L.Mayaram [7] have proposed an elegant homotopy method combining with the shooting method, which can be efficiently applied to the analysis of high Q oscillators. Of course, the harmonic balance method [8] is widely used to the analysis of oscillator circuits. The relaxation method in the frequency domain [9] is also efficiently applied to the weakly nonlinear large scale oscillators.

In this paper, we propose a reactance oscillator which consists of the Cauer circuit coupled with a negative resistance. We can easily synthesize the Cauer circuit by specifying the resonant and anti-resonant frequencies. Then, the oscillator may have multiple oscillations around the anti-resonant frequency points [10]. Unfortunately, since the above mentioned techniques can be only applied to the analysis with a unique oscillation, they cannot find out the multiple oscillations. Our method is based on the harmonic balance method, whose determining equation is described by a set of algeblaic equation. We show here an elegant method such that the equation can be replaced by the equivalent coupled cosine and sine resistive circuit. Solving the circuit with STC based on the Newton homotopy method [11], we can find the multiple oscillations on the solution curve. The analysis can be easily carried out with the transient analysis of SPICE. We will show an interesting example in section 4.

2. Basic algorithm

In order to show the ideas of our algorithm, let us consider a reactance oscillator with a negative resistance as given by Fig.1.

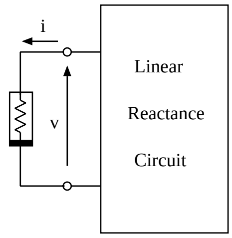


Fig.1 A reactance oscillator

Assume the nonlinear characteristic of the negative resistance is given in the form of power series as follow [12]:

$$i = c_0 + c_1 v + c_2 v^2 + \cdots \tag{1}$$

We also assume the waveform in the following form;

$$v = V_0 + V_1 \cos \omega t + \sum_{k=2}^{K} [V_{2k-2} \cos k\omega t + V_{2k-1} \sin k\omega t] \tag{2}$$

Observe that the waveform does not contain $\sin \omega t$-component because of the autonomous system. Applying the harmonic balance method to the circuit equation shown by Fig.1, we have the *determining equation* as follows:

$$\left.\begin{array}{ll} F_0(V_0, V_1, \cdots, V_{2K-1}, \omega) = 0 & \cdots \text{ DC} \\ F_1(V_0, V_1, \cdots, V_{2K-1}, \omega) = 0 & \cdots \cos \omega t \\ F_2(V_0, V_1, \cdots, V_{2K-1}, \omega) = 0 & \cdots \sin \omega t \\ \cdots\cdots & \\ F_{2K-1}(V_0, V_1, \cdots, V_{2K-1}, \omega) = 0 & \cdots \cos K\omega t \\ F_{2K}(V_0, V_1, \cdots, V_{2K-1}, \omega) = 0 & \cdots \sin K\omega t \end{array}\right\} \tag{3}$$

where ω is the fundamental frequency component to be determined and K is the highest component. Of course, we may have the different frequency component depending on the circuit composition. Thus, the determining equation is described by a set of algebraic equations consisted of $(2K + 1)$-variables and the same number of equations. Note that it is not easy to solve the nonlinear equation because it may have many solutions for the multiple oscillator circuit. For simplicity, set

$$f(v) = 0, \quad v \in R^{2K+1}, \quad f(.) : R^{2K+1} \mapsto R^{2K+1} \tag{4}$$

where $v = [V_0, V_1, \cdots, V_{2K-1}, \omega]^T$. Applying the *Newton homotopy method* [11] to (4), we have the following relation:

$$F(v, \rho) = f(v) + (\rho - 1) f(v_0) = 0 \tag{5}$$

where ρ is an additional variable, and v_0 is an initial guess. Thus, the number of variables is $2K + 2$. Then, the solution is given by the solution curve in the $(2K+2)$-space. The curve can be traced by the application of the *arc-length method* as follows:

$$\left.\begin{array}{l} F(v,\rho) = 0 \\ \displaystyle\sum_{i=1}^{2K+1} \left(\frac{dv_i}{ds}\right)^2 + \left(\frac{d\rho}{ds}\right)^2 = 1 \end{array}\right\}. \tag{6}$$

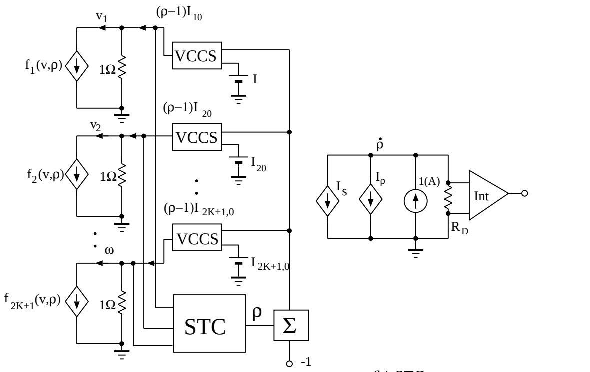
The algebraic-differential equation can be solve by the *backward-differenece method* [13], where the derivative of the kth-oder formula at $s = s^{j+1}$ with the step size h is given by

$$\left.\frac{dv_i}{ds}\right|_{s=s^{j+1}} = \frac{\alpha_{k0}}{h} v_i^{j+1} + Q_{k,i}\left(v_i^j \cdots v_i^{j-k+1}\right) \tag{7}$$

and

$$Q_{k,i} = \frac{1}{h}\left(\alpha_{k,1} v_i^j + \alpha_{k,2}^{j+1} + \cdots + \alpha_{k,k} v_i^{j-k+1}\right).$$

Thus, (6) is again transformed into a set of nonlinear equations, and it can be solved by the Newton-Raphson method, repeatedly. This algorithm is exactly equal to the transient analysis of SPICE, so that we can use SPICE for the curve tracing algorithm.



(a) Circuit diagram (b) STC

Fig.2 Circuit diagram of the Newton homotopy method

The circuit diagram is shown in Fig.2(a) and (b), where "VCCS" is a voltage-controlled current source, and the current sources $f_i(v, \rho)$ are given by

$$\left.\begin{array}{l} f_i(v,\rho) = F_i(v,\rho) - v_i, \quad i = 1, 2, \ldots, 2K+2 \\ I_s = \displaystyle\sum_{i=1}^{2K+1} \left(\frac{dv_i}{ds}\right)^2, \quad I_\rho = \left(\frac{d\rho}{ds}\right)^2. \end{array}\right\} \tag{8}$$

Note that if we assume the voltage difference of $1[\Omega]$ resistance is equal to $-v_i$, then we have

$$-v_i = F_i(v,\rho) - v_i, \quad i = 1, 2, \ldots, 2K+1. \tag{9}$$

which satisfy the relation (5). On the other hand, STC(solution curve tracing circuit) realizes the second term of (6), where R_D is a sufficiently large dummy resistance to avoid a L-J cut set. Furthermore, $I_{10}, I_{20}, \ldots, I_{2K+1,0}$ in the Fig.2(a) denote the initial guesses. Thus, we can easily find the multiple oscillations at $\rho = 1$ by the use of SPICE.

3. Equivalent circuit models of the determining equation

Oscillator circuits usually consists of many linear and/or nonlinear reactive and resistive elements. In order to solve an oscillator with SPICE mentioned in the above, the determining equations should be described in the functional forms. If the nonlinear characteristics are described by the power series function, the Fourier coefficients can be written in the functional forms. However, there are many kinds of nonlinear elements described by the exponential functions, piecewise continuous functions, rational functions and so on. In these cases, the Fourier coefficients cannot be described by the functional forms, and the above mention technique can no longer be applied. Fortunately, for the weakly nonlinear elements, their characteristics can be approximated by the power series [12], then, we can also apply the above technique.

In this section, we propose the cosine and sine circuits corresponding to the determining equation, which is given by the coupled resistive circuits. Each subcircuit has the same structure as the original one except that the reactance elements are replaced by the voltage-controlled current sources or current-controlled voltage sources. Therefore if we use SPICE simulator, the circuit diagram describing the determining equation will be easily drawn on the computer display with SPICE diagram tool. At first, we will show the equivalent circuit models of the nonlinear circuit elements.

3.1 Nonlinear inductor

Assume that the inductor flux is described by the current-controlled characteristic as follows:

$$\hat{\phi}(i_L) = L_1 i_L + L_2 i_L^2 + L_3 i_L^3 + \cdots. \tag{10}$$

Let us consider the Kth higher components of the fundamental frequency component ω, and assume the waveform as follows:

$$i_L = I_{L,0} + \sum_{k=1}^{K} [I_{L,2k-1} \cos k\omega t + I_{L,2k} \sin k\omega t]. \tag{11}$$

Substituting (11) into (10), we have

$$\hat{\phi}(i_L) \cong \Phi_{L,0} + \sum_{k=1}^{K} [\Phi_{L,2k-1} \cos k\omega t + \Phi_{L,2k} \sin k\omega t], \tag{12}$$

where $\Phi_{L,0}, \Phi_{L,1}, \cdots, \Phi_{L,2K}$ are analytical functions of $I_{L,0}, I_{L,1}, \cdots, I_{L,2K}$. Thus, the inductor voltage is given by

$$\hat{v}_L(i_L) = \frac{d\hat{\phi}(i_L)}{dt}$$

$$= \sum_{k=1}^{K} [-k\omega \Phi_{L,2k-1} \sin k\omega t + k\omega \Phi_{L,2k} \cos k\omega t]. \tag{13}$$

Thus, the sine and cosine components for the kth higher harmonic component are respectively given by

$$V_{L,2k} = -k\omega\Phi_{L,2k-1}, V_{L,2k-1} = k\omega\Phi_{L,2k},\ k = 1, 2, \ldots, K \quad (14)$$

where $V_{L,2k-1}$, $V_{L,2k}$ are the amplitudes of the voltages for the kth frequency components. Thus, the inductor is replaced by the coupled current-controlled voltage source as shown in Fig.3.

3.2 Nonlinear capacitor

Assume that the capacitor charge is described by the voltage-controlled characteristic as follows:

$$\hat{q}(v_C) = C_1 v_C + C_2 v_C^2 + C_3 v_C^3 + \cdots. \quad (15)$$

Set the voltage waveform v_C as follows:

$$v_C = V_{C0} + \sum_{k=1}^{K}[V_{C,2k-1}\cos k\omega t + V_{C,2k}\sin k\omega t]. \quad (16)$$

Substituting (16) into (15), we have

$$\hat{q}(v_C) \cong Q_{C0} + \sum_{k=1}^{K}[Q_{C,2k-1}\cos k\omega t + Q_{C,2k}\sin k\omega t], \quad (17)$$

where $Q_{C0}, Q_{C1}, \ldots, Q_{C,2K}$ are analytical function of $V_{C0}, V_{C1}, \ldots, V_{C,2K}$. Then, the voltage-current characteristic of capacitor is given by

$$\hat{i}_C(v_C) = \frac{d\hat{q}(v_C)}{dt}$$

$$= \sum_{k=1}^{K}[-k\omega Q_{C,2k-1}\sin k\omega t + k\omega Q_{C,2k}\cos k\omega t]. \quad (18)$$

Thus, the sine and cosine components for the kth order harmonic are respectively given by

$$I_{C,2k} = -k\omega Q_{C,2k-1}, I_{C,2k-1} = k\omega Q_{C,2k},\ k = 1, 2, \ldots, K \quad (19)$$

where $-k\omega Q_{C,2k-1}, k\omega Q_{C,2k}$ are the currents of the kth frequency components of the nonlinear capacitors. Thus, the capacitor is replaced by the voltage-controlled current sources as shown in Fig.3.

3.3 Nonlinear resistor

There are two types of the voltage-controlled and current-controlled nonlinear resistors. We consider here the voltage-controlled resistor as follows:

$$\hat{i}_G(v_G) = H_{G0} + H_{G1}v_G + H_{G2}v_G^2 + H_{G3}v_G^3 + \cdots. \quad (20)$$

Assume the voltage waveform v_G as follows:

$$v_G = V_{G0} + \sum_{k=1}^{K}[V_{G,2k-1}\cos k\omega t + V_{G,2k}\sin k\omega t]. \quad (21)$$

Substituting (21) into (20), we have

$$\hat{i}_G(v_G) \cong \hat{I}_{G0} + \sum_{k=1}^{K}[\hat{I}_{G,2k-1}\cos k\omega t + \hat{I}_{G,2k}\sin k\omega t]. \quad (22)$$

Thus, the kth order higher harmonic of the sine and cosine are respectively given by

$$I_{G,2k} = \hat{I}_{G,2k}, I_{G,2k-1} = \hat{I}_{G,2k-1},\ k = 1, 2, \ldots, K \quad (23)$$

Note that the voltage-controlled nonlinear resistors are described by the voltage-controlled resistors as shown in Fig.3.

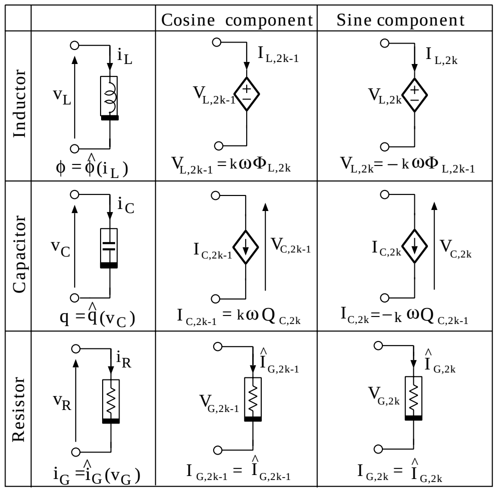


Fig.3 Equivalent cosine and sine circuits

4. An illustrative example

Consider a Cauer oscillator with a negative resistor as shown in Fig.4. Assume that the nonlinear characteristic is given by

$$i_G = -C_1 v_G + C_3 v_G^3,\quad C_1 = 1.\ \ C_3 = 1. \quad (24)$$

At first, we have designed the reactance Cauer such that it has the following resonant and ant-resonant frequencies:

$$\left.\begin{array}{l}\text{Anti}-\text{resonant frequencies}:\ \ \omega_1 = 1,\ \ \omega_3 = 4,\ \ \omega_5 = 6\\ \text{Resonant frequencies}:\ \ \omega_2 = 2,\ \ \omega_4 = 5\end{array}\right\}.$$

Then, we had the following circuit parameters shown in the figure. We also assumed small resistances for all the inductors.

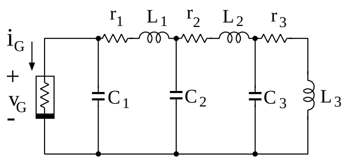


Fig.4 Cauer Oscillator

$$C_1 = 0.1,\quad C_2 = 0.343,\quad C_3 = 0.439,\quad L_1 = 0.417$$
$$L_2 = 0.262,\quad L_3 = 1.058,\quad r_1 = \cdots = r_3 = 0.01$$

Note that since the nonlinear voltage-current characteristic is symmetrical with respect to the origin, we can only consider the odd higher harmonic components. Thus, we assume the waveform v_G as follows:

$$\left.\begin{array}{l}v_G(t) = V_1\cos\omega t + \sum_{k=1}^{K}[V_{2k}\cos(2k+1)\omega t\\ \quad +V_{2k+1}\sin(2K+1)\omega t]\end{array}\right\}. \quad (25)$$

Observe that we have neglected the term of $\sin\omega t$ in (25), because of the autonomous system. In the simulation, although we have chosen $K = 2$, for simplicity, the equivalent circuit for $K = 1$ is given by Fig.5.

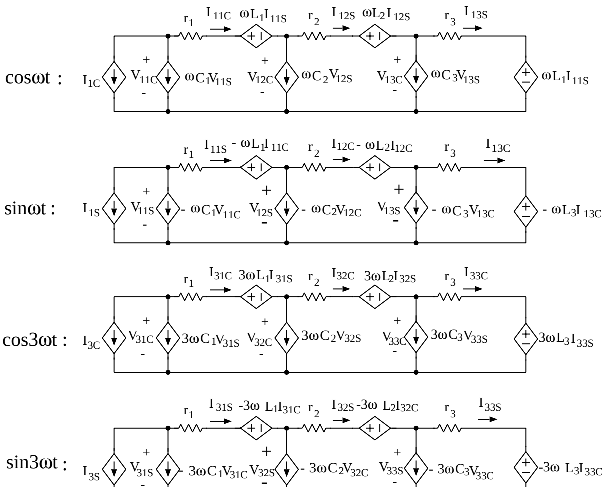


Fig.5 Equivalent circuit describing the determining equation for $K = 1$

Each sub-circuit has the same structure as the original one as shown in Fig.4. The circuit has been solved by the Newton homotopy method shown by Fig.2. The results are shown in Fig.6(a) and (b) for $K = 2$, namely, we have chosen until 5 higher harmonic components. The solutions are obtained at $\rho = 1$. The solutions are also corresponding $\sin\omega t$-component becoming to zero which satisfies the condition (25). Thus, we have obtained 6 solutions as follows:

$$\begin{pmatrix} \omega_1 = 1.0878,\ \omega_3 = 3.6770,\ \omega_5 = 5.722 \text{ (Stable solution)} \\ \omega_0 = 0,\ \omega_2 = 2.0001,\ \omega_4 = 4.999 \quad \text{(Unstable solution)} \end{pmatrix}$$

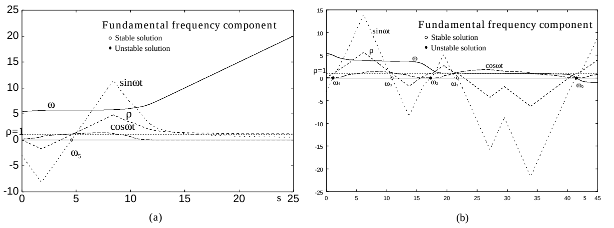


Fig.6(a) Solution curves, $s < 0$, (b) Solution curves, $s > 0$

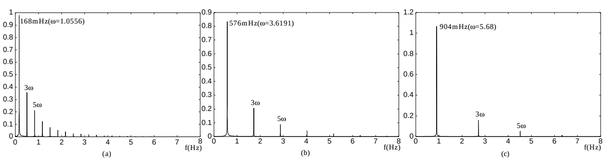


Fig.7 Frequency spectrum.
(a) $\omega = 1.0556$, (b) $\omega = 3.6191$, (c) $\omega = 5.68$

These results have small differences from the results obtained by the FFT analysis to the transient responses, which is shown in Fig.7. The frequency spectrums still have higher harmonic components. Therefore, if we want the exact solutions, we need to consider the more higher harmonic components.

5. Conclusions and remarks

In this paper, we proposed a simple algorithm for calculating multiple oscillations of a reactance oscilator which is based on the SPICE simulator, where we need not to derive the troublesome circuit equation, and to solve the determining equations. We only need to calculate the Fourier coefficients in the functional forms for nonlinear elements, where the nonlinear elements must be described in the form of power series. Then, the equivalent sine and cosine sub-circuits are easily obtained by the use of the SPICE drawing tool. Once the equivalent circuit is drawn, the solutions will be found by the transient tool of SPICE simulator.

For the future problem, we need to apply the algorithm to the practical circuits such as crystal oscillators.

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