FREQUENCY RESPONSE OF NONLINEAR NETWORKS USING CURVE TRACING ALGORITHM

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ABSTRACT

For designing of nonlinear circuits, it is very important to know the frequency response characteristics and the intermodulation. In this paper, we propose an efficient method for calculating the characteristic curves of nonlinear circuits, which is based on the harmonic balance method and a curve tracing algorithm for solving the determining equation. Firstly, applying the harmonic balance method to each element in the circuit, we obtain the determining equation which is realized by two coupled resistive circuits corresponding to the sine and cosine components. Then, the frequency response characteristic curve is calculated by solving the circuit with a STC (solution curve tracing circuit) of Spice.

1. INTRODUCTION

For designing nonlinear circuits, it is very important to calculate the frequency response curves of the circuits and/or the intermodulation characteristics. There have been published many papers concerning to calculate the steady-state response [1]-[5], where the exact waveform is calculated by the iterations only at the specified input frequencies. Therefore, it will takes much computational time to obtain the frequency response characteristic. The Volterra series methods [6]-[7] can be applied to get the frequency responses of weakly nonlinear circuits, where the nonlinear characteristics must be described by the power series expansions. On the other hand, nonlinear circuits sometimes have many interesting phenomena such as *jumping* [8]-[9] and *bifurcation phenomena* [10]. In these phenomena, they may have the multiple solutions at the some regions of input frequency. Unfortunately, these phenomena cannot be solved by the Volterra series method, because it is based on the bilinear operator technique. On the other hand, the harmonic balance method [3]-[5] is widely used for the analysis of nonlinear networks, where the *determining equation* is described by a set of the nonlinear algebraic equations. Although it is usually solved by the Newton-Raphson method, the application will be quite difficult if it has the multiple solutions.

In this paper, we propose an efficient curve tracing algorithm for solving the multiple solutions of determining equation, which is based on the arc-length method [11]. In this case, if we assume the input frequency ω as an additional variable, the solution curve shows the *frequency response characteristics*. In order to develop a user friendly simulator to calculate the frequency response curves, we need to develop the equivalent resistive circuit describing the determining equations. The circuit can be realized by the application of the harmonic balance method to each element, and by replacing it with the equivalent resistive circuit model. In Section 2, we show an algorithm for tracing the frequency response curves, and discuss the stability of the curves. In Section 3, we show the Spice implementation of the determining equations with STC (solution curve tracing circuit), where the frequency response curve is calculated by transient analysis of Spice. The interesting illustrative examples are given in Section 4.

2. CALCULATION OF FREQUENCY RESPONSE CURVE

Now, consider a nonlinear dynamical system driven by a sinusoidal input. Let the circuit equation be described by

$$f(\dot{v}, v, w, \omega t) = 0, \quad f(.) : R^{2n+m} \mapsto R^{n+m} \qquad (1)$$

where, $v \in R^n$ is the state variables, and $w \in R^m$ the non-state variables. Although the steady-state solution will have many higher harmonic components depending on the nonlinearity, we only assume the fundamental frequency component because many of the interesting nonlinear phenomena can be explained by the frequency response characteristic [8-9][1]. Therefore, we set the waveforms as follows:

$$\left. \begin{array}{l} v(t) = V_0 + V_1 \cos \omega t + V_2 \sin \omega t \\ w(t) = W_0 + W_1 \cos \omega t + W_2 \sin \omega t \end{array} \right\} \qquad (2)$$

where V_0 and W_0 are the dc components. Applying the harmonic balance method to (1), we have the following *determining equation*:

$$F(V, W, \omega) = \left[\begin{array}{l} F_0(V_0, V_1, V_2, W_0, W_1, W_2, \omega) \\ F_S(V_0, V_1, V_2, W_0, W_1, W_2, \omega) \\ F_C(V_0, V_1, V_2, W_0, W_1, W_2, \omega) \end{array} \right] = 0 \qquad (3)$$

where the subscripts 0, S and C show the dc, sine and cosine components in the harmonic balance method, respectively. If we assume that ω is an additional variable, then, (3) has $3(n+m)+1$ variables in the $3(n+m)$-dimensional space so that the solution will be given by a solution curve in the hyper-plane. To trace the solution curve, we need to find the initial point on the curve, which is easily calculated by the Newton method and set it as the starting point $(V_0, W_0 \omega_0)$. Now, let us discuss of our curve tracing algorithm.
For simplicity, we set

$$U = (V_0, V_1, V_2, W_0, W_1, W_2, \omega), \quad N = 3(n+m) \qquad (4)$$

[1] Of cause, the method can be extended to the analysis having many higher harmonic components.

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Let us describe a point $U \in R^{N+1}$ on the solution curve by a function of the distance s from the initial point $\mathbf{U}_0$. It satisfies

$$(ds)^2 = \sum_{i=1}^{N+1} (dU_i)^2 \tag{5}$$

in the $N+1$-Euclidean space. Combining (3) with (5), we have the following set of algebraic-differential equations:

$$\left.\begin{array}{c} F_1(U_1, U_2, \ldots, U_{N+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ F_N(U_1, U_2, \ldots, U_{N+1}) = 0 \end{array}\right\} \tag{6.1}$$

$$\sum_{i=1}^{N+1} \left(\frac{dU_i}{ds}\right)^2 = 1 \tag{6.2}$$

It can be efficiently solved by numerical integration formula such as the *backward-difference method* [12], where the derivative of kth order formular at $s = s^{j+1}$ is given by

$$\left.\frac{dU_i}{ds}\right|_{s=s^{j+1}} = \frac{\alpha_{k0}}{h} U_i^{j+1} + Q_{k,i}(U_i\ j, \ldots, U_i^{j-k+1}) \tag{7}$$

$$i = 1, 2, \ldots, N+1$$

Substituting (7) into (6.2) at U^{j+1} point, the algebraic-differential equations are transformed into a set of nonlinear algebraic equations as follows:

$$\left.\begin{array}{c} F_1(U_1^{j+1}, U_2^{j+1}, \ldots, U_{N+1}^{j+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ F_N(U_1^{j+1}, U_2^{j+1}, \ldots, vU_{N+1}^{j+1}) = 0 \\ \displaystyle\sum_{i=1}^{N+1} \left\{\alpha_{k0} U_i^{j+1} + Q(\ldots)\right\}^2 - h^2 = 0 \end{array}\right\} \tag{8}$$

The solution at $s = s^{j+1}$ can be solved by the Newton-Raphoson method, whose Jacobian matrix is given by

$$J(U^j) = \left[\begin{array}{cccc} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} & \cdots & \frac{\partial F_1}{\partial U_{N+1}} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} & \cdots & \frac{\partial F_2}{\partial U_{N+1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_N}{\partial U_1} & \frac{\partial F_N}{\partial U_2} & \cdots & \frac{\partial F_N}{\partial U_{N+1}} \\ P_1(U_1) & P_2(U_2) & \cdots & P_{N+1}(U_{N+1}) \end{array}\right]_{U=U^j} \tag{9}$$

where

$$P_i(U_i) = 2\alpha_{k0}[\alpha_{k0} U_i + Q_{k,i}(U_i^j, \ldots, U_i^{j-k+1})]$$

It is known that the algorithm can trace the solution curve satisfying (6.1) only if $J(U^j)$ is nonsingular; i.e. it means that the rank of $N \times (N+1)$ Jacobian matrix given by (6.1) is always N on the solution curve [11].

Next, let us discuss the **stability** of the solution curves. Set the variations from the steady-state solution as follows:

$$v(t) = (V_0 + \Delta V_0) + (V_1 + \Delta V_1)\cos\omega t + (V_2 + \Delta V_2)\sin\omega t \tag{10.1}$$

$$w(t) = (W_0 + \Delta W_0) + (W_1 + \Delta W_1)\cos\omega t + (W_2 + \Delta W_2)\sin\omega t \tag{10.2}$$

where we assume that the variational value $\{\Delta V_0,\ \Delta V_1,\ \Delta V_2\}$ and $\{\Delta W_0,\ \Delta W_1,\ \Delta W_2\}$ are sufficient small, and slowly varying functions. Thus, we have

$$\left.\begin{array}{l} \Delta v(t) = \Delta V_0 + \Delta V_1 \cos\omega t + \Delta V_2 \sin\omega t \\ \Delta w(t) = \Delta W_0 + \Delta W_1 \cos\omega t + \Delta W_2 \sin\omega t \\ \Delta \dot{v}(t) = \Delta \dot{V}_0 + (\Delta \dot{V}_1 + \omega \Delta V_2)\cos\omega t \\ \qquad\qquad + (\Delta \dot{V}_2 - \omega \Delta V_1)\sin\omega t \end{array}\right\} \tag{11}$$

On the other hand, we have the following variational equation from (1):

$$\left.\frac{\partial f}{\partial \dot{v}}\right|_{u=\hat{u}} \Delta\dot{v} + \left.\frac{\partial f}{\partial v}\right|_{u=\hat{u}} \Delta v + \left.\frac{\partial f}{\partial w}\right|_{u=\hat{u}} \Delta w = 0 \tag{12}$$

at the solution point $\hat{U} = (\hat{V}_0, \hat{V}_1, \hat{V}_2, \hat{W}_0, \hat{W}_1, \hat{W}_2)$.
By the application of the harmonic balance method to (12), we have the following form:

$$A_1 \Delta\dot{V} + A_2 \Delta V + A_3 \Delta W = 0 \tag{13}$$

where

$$\Delta V \equiv [\Delta V_0, \Delta V_1, \Delta V_2],\ \ \Delta W \equiv [\Delta W_0, \Delta W_1, \Delta W_2] \tag{14}$$

On the other hand, we have the variational equation to (3) for a fixed ω in the following form;

$$B_1 \Delta V + B_2 \Delta W = 0 \tag{15}$$

Therefore, from (13) and (15), we have the following characteristic equation:

$$\det \left| \begin{array}{cc} \lambda A_1 + A_2 & A_3 \\ B_1 & B_2 \end{array} \right| = 0 \tag{16}$$

Thus, the stability can be decided by the characteristic roots of the equation. Namely, the steady-state solution will be stable only if all the roots of (16) have the negative real parts. Otherwise, it is unstable.

3. EQUIVALENT CIRCUIT MODELS OF DETERMINING EQUATION

Assume a circuit consisted of linear and nonlinear inductors, capacitors and resistors whose characteristics are approximately described by power expansions. It may also contain linear controlled sources such as voltage controlled current sources, etc. Then, applying the harmonic balance method to each component, we show that it can be replaced by the equivalent circuits corresponding to the *resistive sine and cosine components.*

3.1 Equivalent circuit of nonlinear inductor

Assume that inductor is a current controlled characteristic as follows:

$$\hat{\phi}(i_L) = L_1 i_L + L_2 i_L^2 + L_3 i_L^3 + \cdots \tag{17}$$

Set the current waveform i_L as follow[2]:

$$i_L = I_{L1}\cos\omega t + I_{L2}\sin\omega t \tag{18}$$

Substituting (18) into (17), we have

$$\hat{\phi}(i_L) = \Phi_{L0}(I_{L1}, I_{L2}) + \Phi_{L1}(I_{L1}, I_{L2})\cos\omega t$$

$$+\Phi_{L2}(I_{L1}, I_{L2})\sin\omega t + \Phi_{L3}(I_{L1}, I_{L2}, \omega t) \tag{19}$$

where $\Phi_{L0}(I_{L1}, I_{L2})$, $\Phi_{L1}(I_{L1}, I_{L2})$, $\Phi_{L2}(I_{L1}, I_{L2})$ are amplitudes of the dc and fundamental frequency components.

[2] For simplicity, we assume that the inductor currents, capacitor voltages and resistive voltages do not contain the dc components, and only consider the sinusoidal output responses.

$\Phi_{L3}(I_{L1}, I_{L2}, \omega t)$ is the other frequency components. Therefore, the voltage of inductor is given by

$$\hat{v}_L(i_L) = \frac{d\hat{\phi}(i_L)}{dt}$$

$$= -\omega\Phi_{L1}(I_{L1}, I_{L2})\sin\omega t + \omega\Phi_{L2}(I_{L1}, I_{L2})\cos\omega t$$

$$+\frac{d}{dt}\Phi_{L3}(I_{L1}, I_{L2}, \omega t) \tag{20}$$

Thus, the fundamental frequency components of the sine and cosine are respectively given by

$$V_{L1} = \omega\Phi_{L2}(I_{L1}, I_{L2}),\ V_{L2} = -\omega\Phi_{L1}(I_{L1}, I_{L2}) \tag{21}$$

where V_{L1}, V_{L2} are the amplitudes of the fundamental frequency components of the nonlinear inductor voltage. Thus, the inductor is replaced by the current controlled resistors as shown in Fig. 1.

3.2 Equivalent circuit of nonlinear of capacitor

Assume that capacitor is the voltage controlled characteristic:

$$\hat{q}(v_C) = C_1 v_C + C_2 v_C^2 + C_3 v_C^3 + \cdots \tag{22}$$

Set the voltage waveform v_C as follow:

$$v_C = V_{C1}\cos\omega t + V_{C2}\sin\omega t \tag{23}$$

Substituting (23) into (22), we have

$$\hat{q}(v_C) = Q_{C0}(V_{C1}, V_{C2}) + Q_{C1}(V_{C1}, V_{C2})\cos\omega t$$

$$+Q_{C2}(V_{C1}, V_{C2})\sin\omega t + Q_{3C}(V_{C1}, V_{C2}, \omega t) \tag{24}$$

where $Q_{C0}(V_{C1}, V_{C2})$, $Q_{C1}(V_{C1}, V_{C2})$, $Q_{C2}(V_{C1}, V_{C2})$ are amplitudes of the DC and fundamental frequency components. $Q_{C3}(V_{C1}, V_{C2}, \omega t)$ is other frequency components. Therefore, the current of capacitor is given by

$$\hat{i}_C(v_C) = \frac{d\hat{q}(v_C)}{dt}$$

$$= -\omega Q_{C1}(V_{C1}, V_{C2})\sin\omega t + \omega Q_{C2}(V_{C1}, V_{C2})\cos\omega t$$

$$+\frac{d}{dt}Q_{C3}(V_{C1}, V_{C2}, \omega t) \tag{25}$$

Thus, the fundamental frequency components of the sine and cosine are respectively given by

$$I_{C1} = \omega Q_{C2}(V_{C1}, V_{C2}),\ I_{C2} = -\omega Q_{C1}(V_{C1}, V_{C2}) \tag{26}$$

where Q_{C1}, Q_{C2} are the amplitudes of the fundamental frequency components of the nonlinear capacitors. Thus, the capacitor is replaced by the voltage controlled resistors as shown in Fig. 1.

3.3 Equivalent circuit of nonlinear of resistor

Although there are two types nonlinear resistors of voltage-controlled and current-controlled, we here consider the voltage-controlled resistor:

$$\hat{i}_G(v_G) = A_1 v_G + A_2 v_G^2 + A_3 v_G^3 + \cdots \tag{27}$$

Assume the voltage waveform v_G as follow:

$$v_G = V_{G1}\cos\omega t + V_{G2}\sin\omega t \tag{28}$$

Substituting (28) into (27), we have

$$\hat{i}_G(v_G) = I_{G0}(V_{G1}, V_{G2}) + I_{G1}(V_{G1}, V_{G2})\cos\omega t$$

$$+I_{G2}(V_{G1}, V_{G2})\sin\omega t + I_{G3}(V_{G1}, V_{G2}, \omega t) \tag{29}$$

Thus, the fundamental frequency components of the sine and cosine are respectively given by

$$I_{G1} = I_{G1}(V_{G1}, V_{G2}),\ I_{G2} = I_{G2}(V_{G1}, V_{G2}) \tag{30}$$

Note that the nonlinear resistors do not depend on the frequency. The equivalent circuit is shown in Fig. 1.

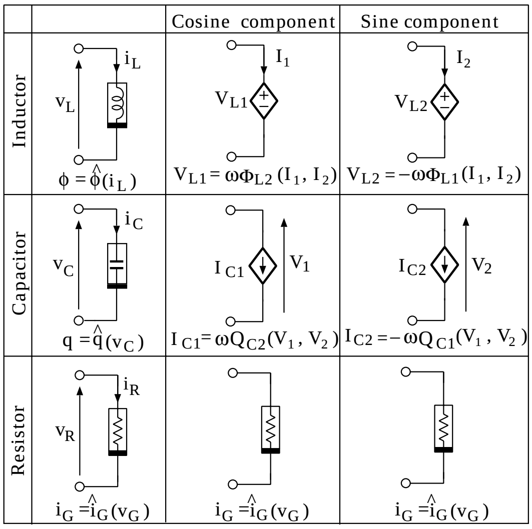


Fig. 1: Equivalent sine and cosine circuit models

Thus, replacing all the elements by the equivalent circuits, we have the *coupled sine and cosine circuits* satisfying the determining equation (3). Since Spice have many useful *analog behavior models* such as voltage controlled current source and polynomial functions and so on, the circuit model can be easily realized with Spice simulator.

4. AN ILLUSTRATIVE EXAMPLE

As an illustrative example, we consider the frequency response of a RLC-ladder circuit having nonlinear capacitors shown in Fig. 2, where the nonlinear capacitors are voltage-controlled, and the sinusoidal input voltage is given by

$$q_1 = C_{11}v_1 + C_{13}v_1^3, \quad q_2 = C_{21}v_2 + C_{23}v_2^3 \tag{31.1}$$

$$e(t) = E_0\cos\omega t \tag{31.2}$$

Applying the above algorithm to the circuit elements, we derive the equivalent circuit models of inductors and capacitors. Set the waveforms of capacitor voltages and inductor currents as follows:

$$v_{Cj} = V_{j1}\cos\omega t + V_{j2}\sin\omega t, \quad j = 1, 2 \tag{33.1}$$

$$i_{Lk} = I_{k1}\cos\omega t + I_{k2}\sin\omega t, \quad k = 1, 2, 3 \tag{33.2}$$

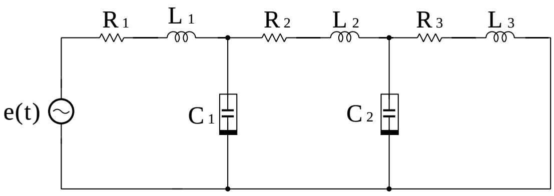
We get the the following cosine and sine components with the harmonic balance method; i.e.

$$I_{C1j} = \omega\left(C_{j1}V_{j1} + \frac{3}{4}C_{j3}V_{j1}(V_{j1}^2 + V_{j2}^2)\right) \tag{32.1}$$

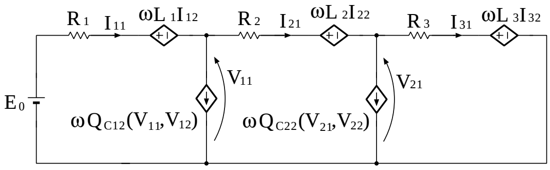
$$I_{C2j} = -\omega\left(C_{j1}V_{j2} + \frac{3}{4}C_{j3}V_{j2}(V_{j1}^2 + V_{j2}^2)\right) \tag{32.2}$$

On the other hand, we have for the linear inductors

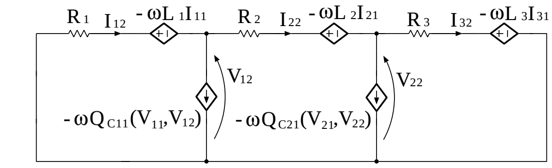
$$V_{Lk1} = \omega L_k I_{k2}, \quad V_{Lk2} = -\omega L_k I_{k1} \tag{33}$$



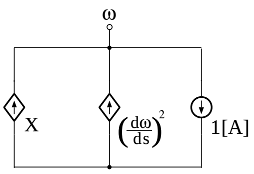
(a) Nonlinear ladder circuit.



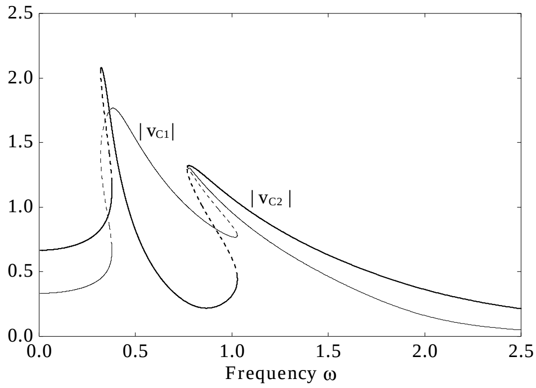
(b) Equivalent circuit of cosine componen



(c) Equivalent circuit of sine componen



(d) STC



(e) Frequency response of ladder circuit.

$$(E_0 = 1.0, C_{11} = C_{12} = 1.0, C_{13} = C_{23} = 3.$$
$$L_1 = L_2 = L_3 = 1.0, R_1 = R_2 = R_3 = 0.1)$$

Thus, every element is transfered to the equivalent sine and cosine components. Thus, the equivalent resistive circuit can be obtained as shown in Fig. 2(b),(c) whose two sub-circuits are coupled in the nonlinear relations in each other. STC in Fig. 2 (d) realize the arc-length method, where it satisfies

$$X = \left(\frac{dV_{11}}{ds}\right)^2 + \left(\frac{dV_{12}}{ds}\right)^2 + \left(\frac{dV_{21}}{ds}\right)^2 + \left(\frac{dV_{22}}{ds}\right)^2 + \left(\frac{dI_{11}}{ds}\right)^2$$
$$+ \left(\frac{dI_{12}}{ds}\right)^2 + \left(\frac{dI_{21}}{ds}\right)^2 + \left(\frac{dI_{22}}{ds}\right)^2 + \left(\frac{dI_{31}}{ds}\right)^2 + \left(\frac{dI_{32}}{ds}\right)^2 \quad (34)$$

$$X + \left(\frac{d\omega}{ds}\right)^2 = 1 \quad (35)$$

The output ω in Fig. 2(d) is inputted in (b) and (c), respectively. Thus, the response curve is traced by the transient analysis of Spice. The dotted lines are unstable solutions. It has jump phenomena.

5. CONCLUSIONS AND REMARKS

We have developed a user friendly simulator tracing frequency response curves, which will be used for simulating nonlinear circuits. At first, we apply the harmonic balance method to each element, and construct the equivalent sine and cosine circuits. It is efficiently solved by the arc-length method with Spice. In future problem, we are going to develop a Spice-oriented simulator of the RF intermodulation.

References

[1] T. Matsuda, Y.Nishio, Y.Yamagami and A.Ushida, “ Steady-state response of nonlinear circuits containing parasitic elements,” *IEICE Trans. :Fundamentals*, vol.E83-A, pp.1023-1031, 2000.

[2] K.Kundert, J.White and A.Sangiovanni-Vincentelli, *Steady-State Methods for Simulating Analog and Microwave Circuits,* Kluwer Academic Publishers, Boston, 1990.

[3] R. Telichevesky, K.Kundert and I.El-Fadel, J.White, “Fast simulation algorithm for RF circuits,” *CICC'96*, 1996.

[4] H.G.Brachtendorf, G.Welsch and R.Laur, “Fast simulation of the steady-state of circuits by harmonic balance technique,” *Proc. of IEEE Int. Symp. on Circuits and Systems,* pp.1388-1391, 1995.

[5] Y.Yamagami, Y.Nishio, A.Ushida, M.Takahashi and K.Ogawa, “Analysis of communication circuits based on multidimensional Fourier transformation,” *IEEE Trans. on CAD,* vol.18, pp.1165-1177, 1999.

[6] Martin Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems*, John Wiley and Sons, 1980.

[7] D.D.Weiner and J.E.Spina, *Sinusoidal Analysis and Modeling of Weakly Nonlinear Circuit: with Application to Nonlinear Interference Effects,* Van Nostrand Reinhold, 1980.

[8] Chihiro Hayashi, *Nonlinear Oscillations in Physical Systems,* McGraw-Hill, 1979.

[9] E.Ikeno and A.Ushida, “The arch-length method for the computation of the characteristic curves,” *IEEE Trans. on Circuits and Systems*, CAS-23, pp.181-183, 1976.

[10] H.Kawakami, “Bifurcation of periodic responses in forced dynamic nonlinear circuit: computation of bifurcation values of the system parameters,” *IEEE Trans. on Circuits and Systems*, CAS-31, pp.248-260, 1984.

[11] A.Ushida and L.O.Chua, “Tracing solution curves of nonlinear equations with sharp turning points,” *Int. Jour. of Circuit Theory and Applications*, pp.1-21, 1984.

[12] L.O.Chua and P-M Lin, *Computer Aided Analysis of Electronic Circuit: Algorithm and Computational Techniques*, Prentice-Hall, 1975.