Detailed Analysis on Phase-Waves Generated in a Ladder of Coupled van der Pol Oscillators

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Abstract— In this study, wave propagation phenomena of phase differences observed in van der Pol oscillators coupled by inductors as a ladder are investigated. For the case of 17 oscillators, interesting phenomena are found. It is called "phase-wave." The wave changing behavior from in-phase to anti-phase or anti-phase to in-phase is called "phase-inversion-wave." We observe that the behavior of the phase-waves generated by giving phase difference of plus value is different from those generated by giving phase difference of minus value. We analyze the difference in detail.

I. INTRODUCTION

Large number of coupled limit-cycle oscillators are useful as models for a wide variety of systems in natural fields, for example, diverse physiological organs including gastrointestinal tracts and axial fiber of nervous systems, convecting fluids, arrays of Josephson junctions and so on. Hence, it is very important to analyze synchronization and the related phenomena observed in coupled oscillators in order to clarify mechanisms of generations or in order to control the generating-conditions of various phenomena in such natural systems. In the field of the electrical engineering, a lot of studies on synchronization phenomena of coupled van der Pol oscillators have been carried out up to now.

Recently, we have discovered very interesting wave propagation phenomena of phase states between two adjacent oscillators in an array of van der Pol oscillators coupled by inductors [1]. In the study, we named the continuously existing wave of changing phase states between two adjacent oscillators from in-phase to anti-phase or from anti-phase to in-phase as "phase-inversion-wave" and explained their generation mechanisms by using the relations between the oscillation frequencies and the phase differences of two adjacent oscillators. We also investigated what kind of phenomena were observed when two phase-inversion-waves collide in the middle of the array [2]. The observed phenomena was classified into three types; completely extinction, completely reflection and intermediate complex phenomenon. Further, the phenomena related with the reflection and the penetration of the phase-inversion-waves in the system including discontinuity were investigated [3]. Behavior of the phase-inversion-waves around a discontinuity were classified into eight types by computer simulations.

In this study, we report the detail on "phase-wave," which is propagation of the phase difference less than 180 degrees between two adjacent oscillators [4]. The phase-waves exist only in the transient states unlike the phase-inversion-waves. In the computer calculations, we produce phase-waves as follows. Almost same initial conditions are given for all oscillators to produce complete in-phase synchronization, which is one stable steady states in the system. After the system settles in the complete in-phase synchronization, the voltage and the current of the arbitrary phase difference is input in the first oscillator. We observe that the behavior of the phase-waves generates by giving phase difference of plus value is different from that generates by giving phase difference of minus value. We analyze the difference in detail.

II. CIRCUIT MODEL

Circuit model used in this study is shown in Fig. 1. N van der Pol oscillators are coupled by coupling inductors L_0. We carried out computer calculations for the cases of $N = 17$. Although we introduce the results only for 17 oscillators, the similar phenomena are observed from both of even and odd. In the computer calculations, we assume the $v-i$ characteristics of the nonlinear negative resistors in the each circuit as the following function.

$$i_r(v_k) = -g_1 v_k + g_3 v_k^3 \quad (g_1, g_3 > 0) \tag{1}$$

The circuit equations governing the circuit in Fig. 1 are written as

[First Oscillator]

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= -x_1 + \alpha(x_2 - x_1) + \varepsilon\left(y_1 - \frac{1}{3}y_1^3\right) \end{aligned} \tag{2}$$

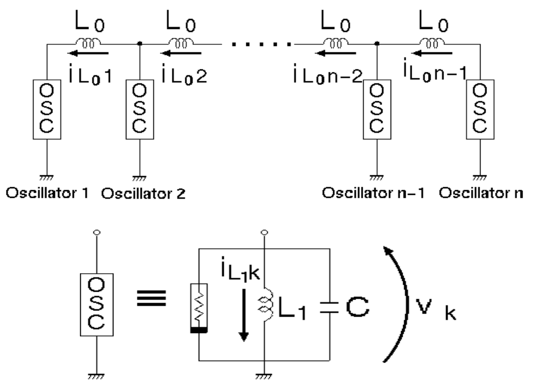

Figure 1: Coupled van der Pol oscillators as a ladder.

[Middle Oscillators]

$$\dot{x}_k = y_k \tag{3}$$
$$\dot{y}_k = -x_k + \alpha(x_{k+1} - 2x_k + x_{k-1}) + \varepsilon\left(y_k - \frac{1}{3}y_k^3\right)$$
$$(k = 2 \sim N{-}1)$$

[Last Oscillator]

$$\dot{x}_N = y_N \tag{4}$$
$$\dot{y}_N = -x_N + \alpha(x_{N-1} - x_N) + \varepsilon\left(y_N - \frac{1}{3}y_N^3\right)$$

where

$$t = \sqrt{L_1 C}\tau \ , \ i_{L_1 k} = \sqrt{\frac{Cg_1}{3L_1 g_3}}x_k \ , v_k = \sqrt{\frac{g_1}{3g_3}}y_k \ ,$$
$$\alpha = \frac{L_1}{L_0} \ , \ \varepsilon = g_1\sqrt{\frac{L_1}{C}} \ , \ \frac{d}{d\tau} = \text{“}\cdot\text{”} \ . \tag{5}$$

It should be noted that α corresponds to the coupling of oscillators and ε corresponds to the nonlinearity of oscillators. Simulation are calculated by using the fourth-order Runge-Kutta method with Eqs. (2)–(4).

III. PHASE-WAVES

A. PHASE-WAVES

In this section, wave propagation phenomenon observed from the circuit with 17 oscillators is investigated. Further, we fix $\alpha = 0.050$ and $\varepsilon = 0.30$.

Throughout the paper, we define the phase difference between two adjacent oscillators as follows:

$$\Phi_{k,k+1}(n) = \frac{\tau_k(n) - \tau_{k+1}(n)}{\tau_k(n) - \tau_k(n-1)} \times \pi$$

where $\tau_k(n)$ is time when the voltage of OSCk crosses 0[V] at n-th time.

Fig. 3 show typical examples of observed phase-waves which are propagation of the phase-difference between adjacent oscillators. These results are obtained for the same parameter values by changing initial conditions as follows:

1. Setting as initial conditions of all oscillators the same.
2. Putting as the arbitrary phase difference of the voltage and the current of one oscillator.

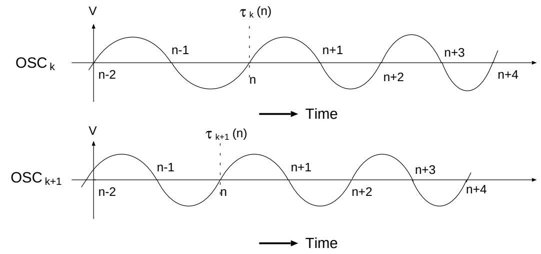

Figure 2: Definition of phase difference.

In upper diagrams of Fig. 3, vertical axis is the sum of voltages of adjacent oscillators, and horizontal axis is time. Hence, the diagrams show how phase differences between adjacent oscillators change as time goes. In lower figures, snapshots of attractor of each oscillator and phase states between adjacent oscillators are shown respectively.

B. CLASSIFICATION

When phase-waves are generated by phase difference of plus value or minus value at an edge of the ladder, observed phenomena are classified into four patterns by input phase difference as follows:

<plus value>

(a) around $0 \sim +69$[degree]: The phase difference decreases while the phase-waves are propagating. The phase-waves become extinction at the end of the array.

(b) around $+70$[degre]]: The phase difference increases while the phase-waves are propagating. After the phase-waves reflect at the end of the array, they become extinction while propagating. Because the phase-waves do not have enough phase differences to become anti-phase at the edge of the array.

(c) around $+71 \sim +171$[degree]: The phase difference increases while the phase-waves are propagating. The phase-waves reflect at the edge of the array and change to the phase-inversion-waves, because the phase-waves have the phase differences to become anti-phase at the edge of the array.

(d) around $+172 \sim +180$[degree]: The phase-waves change to the phase-inversion-waves while the phase-waves are propagating.

<minus value>

(a) around $0 \sim -67$[degree]: The phase difference decreases while the phase-waves are propagating. The phase-waves become extinction at the end of the array.

(b) around -68[degree]: The phase difference increases while the phase-waves are propagating.

The phase-waves reflect at the end of the array and change to the phase-inversion-waves, because the phase-waves have the phase differences to become anti-phase at the edge of the array.

(c) around $-69 \sim -180$[degree]: The phase difference increases while the phase-waves are propagating. The phase-waves become the phase-inversion-waves before which arrive the edge of the array.

(d) around $-59 \sim -85$[degree]: Two pair of waves are generated.

We observed that the behavior of the phase-waves generated by giving phase difference of plus value is different from that generated by giving phase difference of minus value. We analyze the difference in detail. Further, when the phase-waves are generated by giving phase difference of minus value, we found the phenomena that phase-waves of two sets generate. Fig. 4 shows domain of the observed phenomena.

IV. CONCLUSION

In this study, the phase-waves were investigated by numerical calculation. We found that the behavior of the phase-waves generated by giving phase difference of plus value is different from that generated by giving phase difference of minus value. We analyze the difference in detail. Further, when the phase-waves are generated by giving phase difference of minus value, we found the phenomena that phase-waves of two sets generate.

References

[1] M.Yamauchi, M.Wada, Y.Nishio and A.Ushida, "Wave Propagation Phenomena of Phase States in Oscillators Coupled by Inductors as a Ladder, " *IEICE Trans. Fundamentals*, vol.E82-A, no.11, pp.2592-2598, Nov. 1999.

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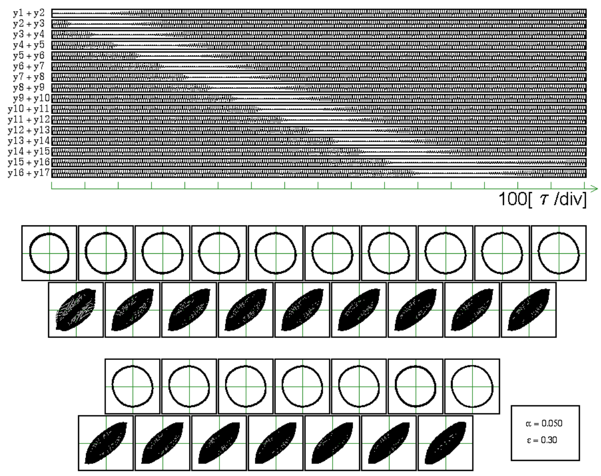
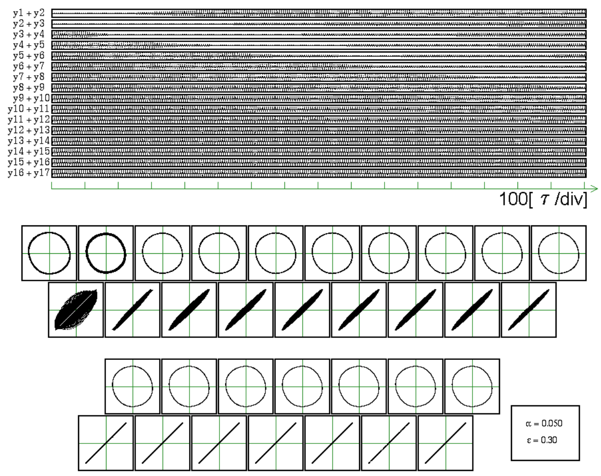
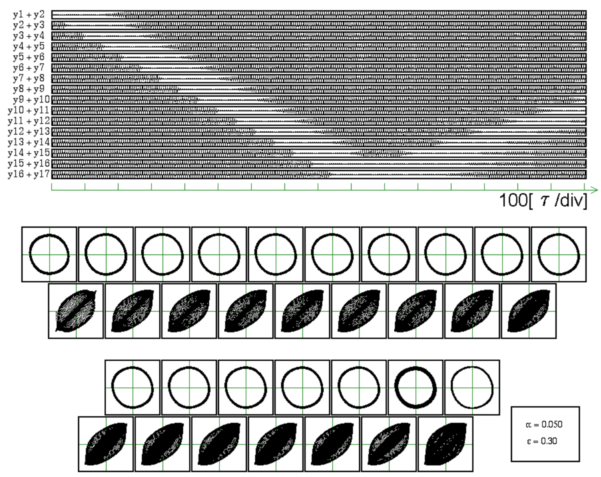


(i) input phase difference +60[deg](domain plus-(a))

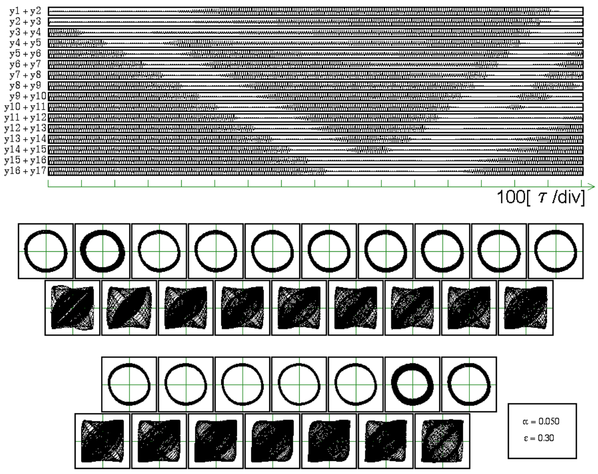


(ii) input phase difference -50[deg](domain minus-(a))

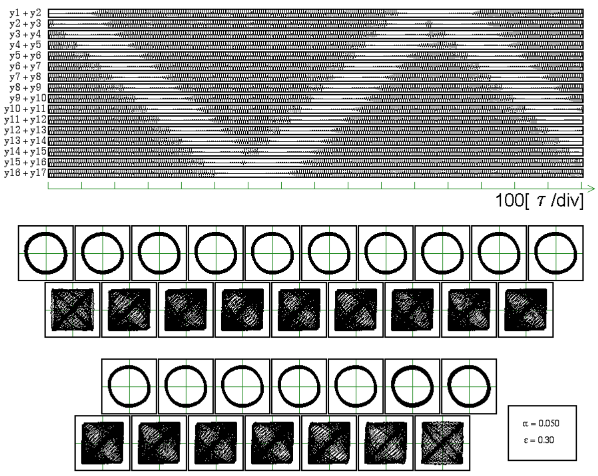


(iii) input phase difference +70[deg](domain plus-(b))

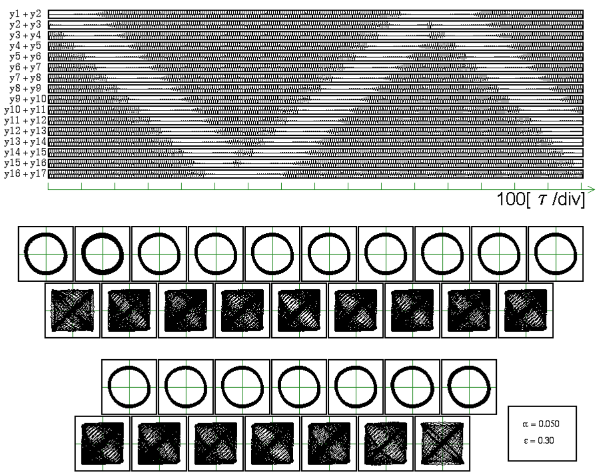
Figure 3: Phase-waves. $\alpha = 0.05, \varepsilon = 0.3$.



(iv) input phase difference −68[deg](domain minus-(b) and minus-(d))

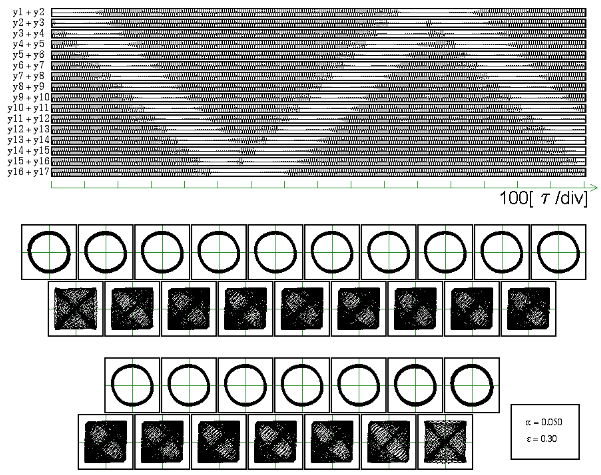


(v) input phase difference +135[deg](domain plus-(c))

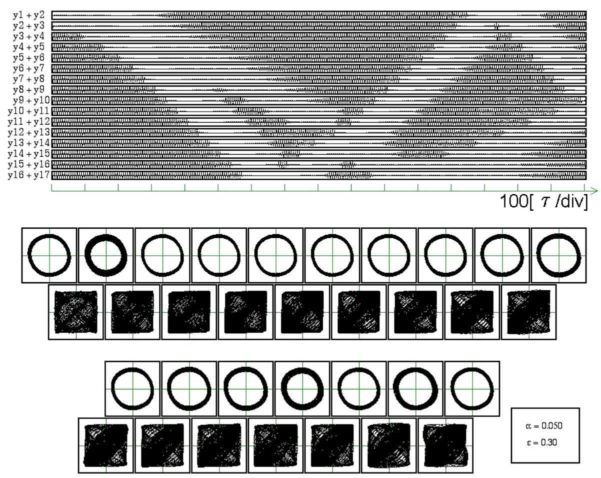


(vi) input phase difference −135[deg](domain minus-(c))

Figure 3: (continued).

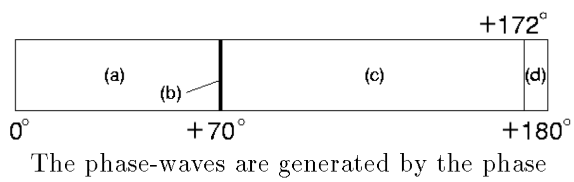


(vii) input phase difference +175[deg](domain plus-(d))

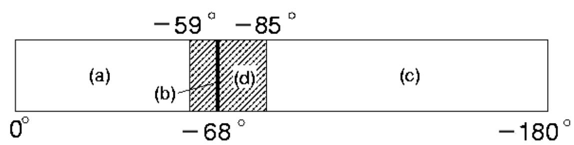


(viii) input phase difference −70[deg](domain minus-(d))

Figure 3: (continued).



The phase-waves are generated by the phase difference of **plus** value.



The phase-waves are generated by the phase difference of **minus** value.

Figure 4: Domain of observed phenomena that are gerated by the phase difference of plus or minus value.