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Characteristic Curve of Nonlinear Resistive Circuits and its Stability

Akio USHIDA, Lingge JIANG and Yoshifumi NISHIO
Department of Electrical and Electronic Engineering, Faculty of Engineering, Tokushima University, Tokushima, 770, Japan
Phone: +81-886-56-7469
Fax. 0886-56-7471
Email: ushida@ee.tokushima-u.ac.jp

Abstract

Distinguishing the stability of characteristic curves for nonlinear resistive circuits is requirement and importance to design various electronic circuit exactly. Since every resistive element has a parasitic component, solutions on the characteristic curves are stable or unstable. In this paper, we show that the stability will be mainly changed at the bifurcation points such as limit point and branch point. Applying the **curve tracing method**, we can decide the unstable regions on characteristic curves by the locations of bifurcation points.

1 Introduction

In this paper, we discuss the stability of the solution curves for nonlinear resistive circuits including parasitic elements. Although the DC solution is determined by analyzing the nonlinear resistive circuit, its equilibrium point will be the one of stable or unstable because every resistive element has small parasitic elements in practice. There are some papers discussing the stability of nonlinear networks. In references [1,2], a globally asymptotically stable condition is discussed for nonlinear dynamic networks in a qualitative manner. In reference [3], a simple technique is proposed to identify unstable DC operating points. Of course, the stability can be decided by solving the variational equation at each equilibrium point obtained by the DC analysis, however, that is very time-consuming. We show here that the stability is *mainly* changed at the bifurcation points such as turning and branch points [4-5] on the DC characteristic curves, so that the stability of the solution curve is easily found by the locations of bifurcation points.

2 Unstable regions of solution curves

Now, consider electronic circuits containing bipolar transistors, FETs and so on. The solution curve can be calculated by solving a resistive circuit composed of n equations in $(n+1)$ variables

$$f(x)=0, \quad f: R^{n+1} \longrightarrow R^n \tag{1}$$

Assume $f(x)$ is C^2 continuous in $x \in R^{n+1}$. Let us describe the variable by $x = x(s)$ as a function of *arc-length* s from the starting point x_0. Then, the solution of (1) satisfies the following set of algebraic-differential equations [7]:

$$f(x)=0 \tag{2.1}$$

$$\left(\frac{dx_1}{ds}\right)^2+\left(\frac{dx_2}{ds}\right)^2+\cdots+\left(\frac{dx_{n+1}}{ds}\right)^2=1 \tag{2.2}$$

Since the solution curve is a continuous function of s even at the *limit point* [5], we have from (2)

$$D\Gamma(x)\begin{pmatrix} \frac{dx_1}{ds} \\ \cdots \\ \frac{dx_n}{ds} \\ \frac{dx_{n+1}}{ds} \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix} \tag{3}$$

where

$$D\Gamma(x)=\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \\ \frac{dx_1}{ds} & \cdots & \frac{dx_n}{ds} & \frac{dx_{n+1}}{ds} \end{pmatrix} \tag{4}$$

Observe that the first $n \times (n+1)$ submatrix corresponds to the Jacobian matrix of $f(x)$, and the last row shows the derivatives of the curve. Our curve tracing algorithm [6] efficiently traces the solution curve satisfying (2) . In this case, it is proved that whenever the rank of Jacobian matrix of $f(x)$ is n, the coefficient matrix $D\Gamma(x)$ is nonsingular, so that we can trace even for the *limit points* [5]. Thus, we have the following relation by the Cramer's formula to $(n+1)$th variable

$$\frac{dx_{n+1}}{ds}=\frac{\det|D_n f(x)|}{\det|D\Gamma(x)|} \tag{5}$$

where $D_n f(x)$ is the Jacobian matrix for the variable $\{x_1, x_2, \ldots, x_n\}$

Now, assume that we have the following dynamic equation by considering the parasitic elements:

$$P\frac{dx}{dt} = f(x), \quad \text{for } P = \begin{pmatrix} C_p & 0 \\ 0 & L_p \end{pmatrix} \tag{6}$$

Then, the variational equation at an equilibrium point x is given by

$$P\frac{d\Delta x}{dt} = D_n f(x)\Delta x \tag{7}$$

Thus, the stability condition of the resistive circuits is decided by the eigenvalues of the Jacobian matrix $D_n f(x)$. We have the following stability property around the *limit point.*

Theorem 1 *If the starting point of solution curve is stable, when the solution curve has passed through a limit point, the stability is changed at the point.*

Proof: The *limit point* is a turning point such that the direction of the solution curve is changed and $dx_{n+1}/ds = 0$ at the point. This means that the sign of $\det|D_n f(x)|$ is changed after passing through the limit point because of the nonsingularity of $D\Gamma(x)$ in (5) [5]. Here, we transform (7) as follows:

$$\frac{d\Delta x}{dt} = P^{-1} D_n f(x)\Delta x$$

The eigenvalues of variational equation satisfy the following relation[7]:

$$\det|P^{-1} D_n f(x)| = \det|P^{-1}| \prod_{i=1}^{n} \lambda_i \tag{8}$$

We assume that $\det|P^{-1}| \neq 0$ holds, so that the stability depends on the eigenvalues of $D_n f(x)$, where $\lambda_i (i = 1, 2, \ldots, n)$ are the eigenvalues composed of real and/or complex conjugates. Thus, the change of sign (5) means that the sign of one of the real eigenvalues is changed after passing through the limit point, so that the stability is changed. ■

Next, we consider the stability of the solution curve around the *branch bifurcation point* [4], where two solution curves cross at a point. It is known that the rank of the Jacobian matrix to (1) for $\{x_1, x_2, \ldots, x_{n+1}\}$

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \end{pmatrix}$$

is reduced to less than n. Hence, the matrix $D\Gamma(x)$ becomes singular at the bifurcation point. We have the following theorem around the point.

Theorem 2 *Let $\Gamma(x)$ be a smooth solution curve passing through the branch bifurcation point. Then, the stability of solution is changed at the point.*

Proof: For simplicity, put

$$d_1(x_{n+1}) \equiv \det|D\Gamma(x)| \tag{9}$$

Now, applying Taylor expansion to $d_1(x_{n+1})$ at two points $x^*_{n+1} - \Delta x_{n+1}$ and $x^*_{n+1} + \Delta x_{n+1}$ before and after the bifurcation point x^*, we have

$$d_1(x^*_{n+1} - \Delta x_{n+1}) = d_1(x^*_{n+1}) - d_1{}'(x^*_{n+1})\Delta x_{n+1} + \cdots \tag{10.1}$$

$$d_1(x^*_{n+1} + \Delta x_{n+1}) = d_1(x^*_{n+1}) + d_1{}'(x^*_{n+1})\Delta x_{n+1} + \cdots \tag{10.2}$$

where $'$ indicates the derivative with respect to x_{n+1}. At the branching point x^*, the following relations hold [4]

$$\text{rank}(Df(x^*)) = n-1, \;\; d_1(x^*_{n+1}) = 0, \;\; d_1{}'(x^*_{n+1}) \neq 0 \tag{11}$$

Multiplying the two equations in (10), we obtain

$$d_1(x^*_{n+1} - \Delta x_{n+1}) d_1(x^*_{n+1} + \Delta x_{n+1}) \cong -[d_1{}'(x^*_{n+1})]^2 \Delta x^2_{n+1}$$

Thus, the sign of the denominator of (4) is changed whenever it passes through the point. We have the same result as $\det|D\Gamma(x)|$ for

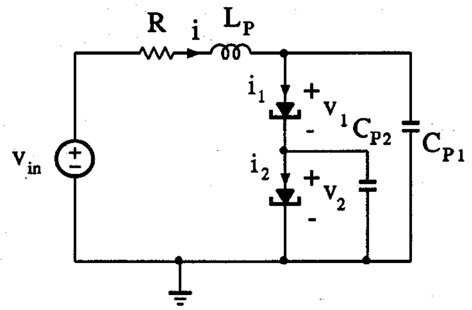
$$d_2(x_{n+1}) \equiv \det|D_n f(x)| \tag{12}$$

because the rank of $Df(x)$ is less than n at the bifurcation point. Thus, the sign of (4) is not changed at the point, so that the direction of solution curve dx_{n+1}/ds is never changed at the branch bifurcation point. But the stability of the solution curve is changed. The instability of the equilibrium point after the bifurcation point will be a saddle type. ■

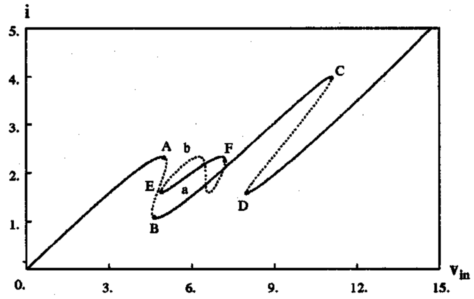
As a special case, there are many symmetric circuits such as Flip-Flop circuit. In this case, they sometimes have an interesting property such that one of the solution curves is *symmetric* with respect to another one. This type of bifurcation is termed as *pitchfork*point [4].

Corollary 1 *At a pitchfork point, one of the solution curves changes the stability at the point, while the others remains the same stability passing through the point. This is because that the pitchfork bifurcation has symmetric solution curves.*

Remark: In *Theorem 1* and 2, the instability regions are determined by investigating whether the



(a)Two-tunnel-diode circuit



(b)Driving-point characteristic curves

Figure 1: Two-tunnel-diode circuit

variational equation has the positive real eigenvalues or not. However, it may sometimes happen that it has the complex conjugate eigenvalues having positive real parts. For this kind of instability, the equilibrium point behaves as an *unstable focal point*, and the sign of dx_{n+1}/ds is not changed at the bifurcation point. This bifurcation is called *Hopf bifurcation.*

3 Illustrative examples

3.1 Two-tunnel-diode circuit [6]

Consider the two-tunnel-diodes circuit shown in Fig.1(a). Let us choose such that the *normal tree* contains v_{in} and two tunnel diodes. Put the parasitic element L_p into the co-tree R and C_{p1}, C_{p2} between the tree diodes and ground.

The circuit equation is given by

$$C_{p1}\frac{dv_{Cp1}}{dt} = i - g_1(v_1)$$

$$C_{p2}\frac{dv_{Cp2}}{dt} = g_1(v_1) - g_2(v_2)$$

$$L_p\frac{di}{dt} = v_{in} - (v_1 + v_2) - Ri$$

where

$$g_1(v_1) \equiv 2.5v_1^3 - 10.5v_1^2 + 11.8v_1$$

$$g_2(v_2) \equiv 0.43v_2^3 - 2.69v_2^2 + 4.56v_2$$

The driving point characteristic for $R = 1.5$, $L_p = 1$ and $C_{p1} = C_{p2} = 1$ is shown in Fig.1(b), where the dotted lines show the unstable regions. Observe that there are small regions of the Hopf bifurcations before and after the $dv_{in}/ds < 0$ regions in the $dv_{in}/ds > 0$. On the other hand, there is a closed loop $(EaFb)$, where the region (EaF) is stable and (FbE) unstable. Note that once the stability is checked at a point on the closed loop, the whole of the stability can be known by *Theorem 1.*

3.2 Hopfield network

Hopfield neural networks are sometimes applied to solve combinatorial problems such as the traveling salesman problem, and the layout of VLSI circuits. Now, consider the circuits containing 6 synapses whose equation is given by [2]

$$\frac{du_i}{dt} = \sum_{j=1}^{6} w_{ij}x_j - \frac{a}{2}\log\frac{x_i}{1-x_i} + I_i$$
$$i = 1, 2, \ldots, 6$$

where

$$W = \begin{pmatrix} 0 & 1 & -2 & -2 & -2 & -2 \\ 1 & 0 & -2 & -2 & -2 & -2 \\ -2 & -2 & 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & -2 & 0 & 1 \\ -2 & -2 & -2 & -2 & 1 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 3.5 & 3.5 & 5.0 & 5.0 & 3.5 & 3.5 \end{pmatrix}^T$$

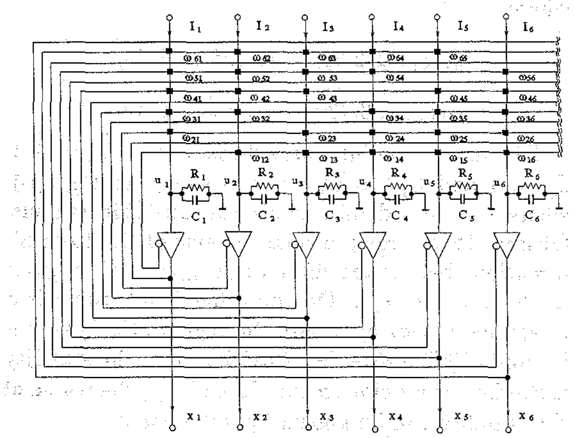
Setting $du_i/dt = 0$, the stationary solutions are obtained. Choosing a as an additional variable, we have a set of 6 algebraic equations with 7 variables. The solution curves are obtained starting from $a = 0.1$ [15]. The curves in the (x_1, x_3, x_7)-plane are shown in Fig.8, where we choose $a = 0.29x_7 + 0.1$. We found 9 pitchfork points and 4 limit points.

Note that since the coefficient matrix W is symmetric, all of the eigenvalues are real, and the equilibrium points belong to the nodal or saddle points. We show the unstable curves by dotted lines. Their stabilities are determined by the application of *Theorem 2* and *Corollary 1.*

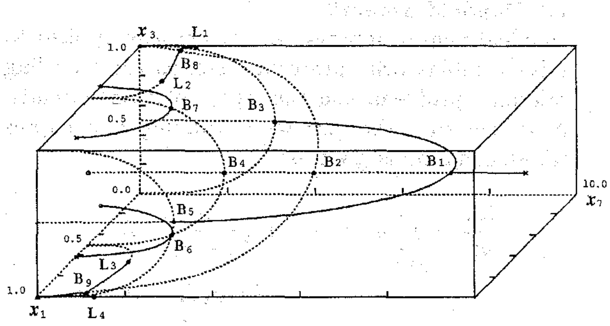
4 Conclusions and remarks

In this paper, the stability of DC solution curves is examined by introducing parasitic elements, such as

[2]The example is given by Prof. A.Sakamoto at Tokushima university.



(a) A Hopfield network



(b) Stability of the solution curve for the Hopfield network

Figure 2: Hopfield network

a small capacitor between every resistor and ground, and inductor in series to every co-tree resistor.

We have proved two theorems and one corollary which are very useful to check the stability of the solution curves. Since the stability will be mainly changed at the bifurcation points such as a limit point and branch bifurcation point, we can know the stability of solution curves without investigating the variational equation.

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