RIGOROUS ANALYSES OF WINDOWS IN A SYMMETRIC CIRCUIT

Yoshifumi Nishio †, Naohiko Inaba †, Shinsaku Mori † and Toshimichi Saito ‡

† Department of Electrical Engineering, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223 Japan
‡ Department of Electrical Engineering, Sagami Institute of Technology
1-1-25 Tsujido-Nishikaigan, Fujisawa, 251 Japan

Abstract

Two types of windows are found in a symmetric circuit, which cannot be seen in a logistic map and are inherent in the symmetric structure of this circuit. The purpose of this paper is to derive Poincaré map strictly as a one-dimensional map from the circuit and to prove rigorously that the two types of windows appear alternately and infinitely inside some windows.

1 INTRODUCTION

Symmetric structures are often observed in chaos generating circuits. "Symmetry" is one of the features which characterize chaos generating circuits. Therefore, it is of great importance to investigate bifurcation phenomena characterized by the symmetry of circuits.

In the meantime, some symmetric one-dimensional maps have been studied. For example, J. Testa et al. have investigated a cubic map and have discovered the two types of windows which cannot be seen in a logistic map [1]: one exhibits complex bifurcation phenomena such as symmetry breaking and symmetry recovering in its own region, while the other is the windows appearing on the condition that one chaotic attractor bifurcates to two distinct periodic attractors. These windows have interesting bifurcation structures and seem to be deeply related to those of symmetric systems.

We find these two types of windows in a symmetric circuit. which are similar to the windows generated in a cubic map [1]. The purpose of this paper is to analyze these windows rigorously by using a degeneration technique as shown in [2][3]. This degeneration means that the diode in the circuit is assumed to operate as an ideal limiter. In this case, Poincaré map is derived strictly as a one-dimensional map. Especially it is proved rigorously that the two types of windows appear alternately and infinitely inside some windows with some reasonable assumptions by utilizing a certain scaling mechanism on the Poincaré map.

2 CIRCUIT MODEL

We consider the circuit in Fig. 1 whose structure is symmetric with respect to the origin. This circuit consists of three memory elements, one linear negative resistance and only one nonlinear resistance which consists of two diodes.

At first, we approximate the $i-v$ characteristic of the nonlinear resistance as the following 3-segment piecewise-linear function as shown in Fig. 2(a).

$$v_d(i_2) = r_d/2\ (|i_2 + V/r_d| - |i_2 - V/r_d|). \qquad (1)$$

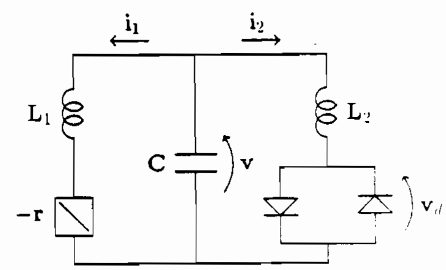


Fig. 1 Circuit model.

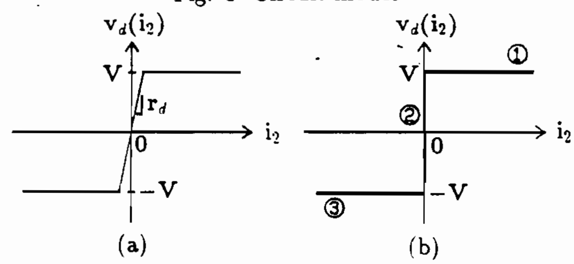


Fig. 2 The $i-v$ characteristic of the nonlinear resistance.

Then the governing equation of the circuit in Fig. 1 is described by a piecewise-linear third order ordinary differential equation.

$$L_1\frac{di_1}{dt} = v + ri_1,\quad L_2\frac{di_2}{dt} = v - v_d(i_2),\quad C\frac{dv}{dt} = -i_1 - i_2. \qquad (2)$$

In order to investigate the phenomena observed from our circuit, we use the degeneration technique, which means that the nonlinear resistance in the circuit operates as an ideal limiter as shown in Fig. 2(b). We have already verified that this method is effective to clarify the mechanism of chaos for a circuit family including one diode [3]. In this case the solution of Eq. (2) seems to be well explained by the following Ideal Model.

[Ideal Model]

① When $y > 0$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -\beta & 0 \end{bmatrix} \begin{bmatrix} x + 1/\alpha \\ y - 1/\alpha\beta \\ z - 1 \end{bmatrix}.$$

② When $y = 0$,

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

③ When $y < 0$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -\beta & 0 \end{bmatrix} \begin{bmatrix} x - 1/\alpha \\ y + 1/\alpha\beta \\ z + 1 \end{bmatrix}, \qquad (3)$$

where

$$i_1 = \sqrt{\frac{C}{L_1}} Vx, \quad i_2 = \frac{\sqrt{L_1 C}}{L_2} Vy, \quad v = Vz, \quad t = \sqrt{L_1 C}\tau,$$

$$r\sqrt{\frac{C}{L_1}} = \alpha, \quad \frac{L_1}{L_2} = \beta, \quad r_d \frac{\sqrt{L_1 C}}{L_2} = \gamma, \quad "\cdot" = \frac{d}{d\tau}, \qquad (4)$$

These equations in the regions ①, ② and ③ are connected by the following transitional conditions.

$$\begin{array}{ll} ① \to ② : \ y = 0, & ② \to ① : \ z = 1, \\ ② \to ③ : \ z = -1, & ③ \to ② : \ y = 0. \end{array} \qquad (5)$$

Since the circuit equation is piecewise-linear, the general solutions in each region can be explicitly given.

Define the subspaces $D^+ : y > 0$, $D^o : y = 0$ and $D^- : y < 0$.

3 POINCARÉ MAP

Figure 3 shows the projection of the vector field onto x-z plane. We define the following subspaces.

$$\begin{aligned} B^{\pm} &= \{(x, y, z) : z = \pm 1, \ y = 0\}, \\ L^+ &= \{(x, y, z) : x > 0, y = 0, z = -0.2\}, \\ L^- &= \{(x, y, z) : x < 0, y = 0, z = 0.2\}. \end{aligned} \qquad (6)$$

The subspaces B^+ means the transitional condition ②→ ① while B^- means the transitional condition ②→ ③ .

First let A be the x-coordinate of the initial point on L^+ whose flow should be tangent to the line B^- at $x = 0$.

We discuss the solution having the initial value at a point on L^+. If the parameters are chosen properly, the solution which leaves the line L^+ hits L^+ again at some time.

1. Case $0 < x_0 \leq A$: The solution rotates divergently around O constrained onto D^o and it hits L^- without reaching the threshold $z = -1$.
2. Case $A < x_0$: The solution hits the line B^- and enters D^- at some time. The solution which entered D^- hits the plane D^o. We consider the case that the point exists in the hatched region which is illustrated in Fig. 3. If the point exists in this region, the solution which is back on D^o starts to rotate divergently around O constrained onto D^o and hits L^-.

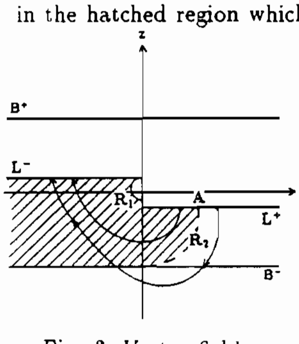


Fig. 3 Vector field.

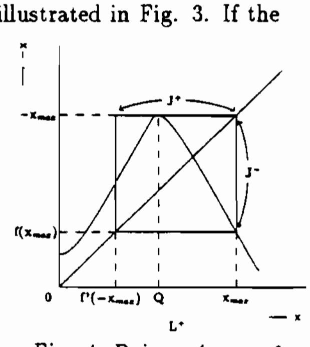


Fig. 4 Poincaré map f.

Therefore, a discrete one-dimensional map which transforms a point on L^+ into a point on L^- can be defined.

$$f \ : \ L^+ \to L^-, \quad x_0 \to f(x_0) \qquad (7)$$

where x_0 is the x-coordinate of the initial point on L^+ and $f(x_0)$ is the x-coordinate of the point on L^- which the solution leaving from L^+ hits L^-. Since Eq. (6) is symmetric, we could obtain the map f' which transforms a point on L^- into a point on L^+. The map f' has the same form as f.

In the following discussions, we fix β at 3 and choose α as a bifurcation parameter.

Figure 4 shows an example of the map f. Let Q be the point on L^+ at which f has an extremum and let $-x_{max}$ be $f(Q)$. Define two intervals $J^+ = [f'(-x_{max}), x_{max}] \subset L^+$ and $J^- = [f(x_{max}), -x_{max}] \subset L^-$ as shown in Fig. 4.

Define

$$F = f' \circ f, \qquad F' = f \circ f'. \qquad (8)$$

Hereafter, we use a superscript "+" on an interval which is a part of J^+ and similarly we use "−" on an interval which is a part of J^-.

Figure 5 shows examples of the windows which are found in the bifurcation diagram on L^+. These windows are similar to those generated in a cubic map. We call the window in Fig. 5(a) as the window of Type 1 and also the window in Fig. 5(b) as the window of Type 2.

3.1 THE WINDOW OF TYPE 1

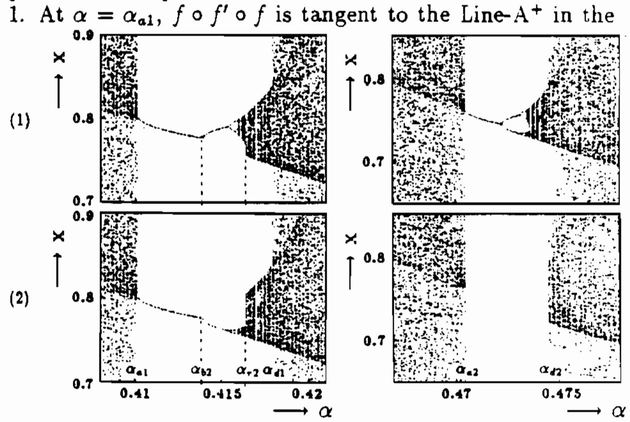
We discuss the mechanism of the window of Type 1.

Let Line-A$^+$, Line-A$^-$, Line-B$^+$ and Line-B$^-$ be the lines which satisfy $f(x) = -x$, $f'(x) = -x$, $F(x) = x$ and $F'(x) = x$, respectively.

Consider the composite map $f \circ f' \circ f$. Figure 6 shows an example of $f \circ f' \circ f$. As α increases, $f \circ f' \circ f(Q)$ descends as shown in Fig. 6. The neighborhood of Q is shown in Fig. 7. As far as we carry out the computer simulations, the following situations exist.

[Situation 1]

1. At $\alpha = \alpha_{a1}$, $f \circ f' \circ f$ is tangent to the Line-A$^+$ in the



(a) (b)

Fig. 5 Windows (1)$x_0 = Q$, (2)$x_0 = -Q$.

neighborhood of Q. F has two periodic points near Q for $\alpha_{a1} < \alpha < \alpha_{max}$. At least for $\alpha_{a1} < \alpha < \alpha_{w1}$ one is stable periodic point and the other is unstable periodic point. Let the former be x_{s1} and also latter be x_{u1}.
2. At $\alpha = \alpha_{w1}$, Q is equal to x_{s1}. That is, $f \circ f' \circ f(Q) = -Q$ is satisfied.
3. At $\alpha = \alpha_{d1}$, $F^3(Q)$ is equal to x_{u1}.

Define the intervals $J_3^+ = [x'_{u1}, x_{u1}] \subset J^+$ and $J_3^- = [-x'_{u1}, -x_{u1}] \subset J^-$ as shown in Fig. 7, where $f \circ f' \circ f(x'_{u1}) = -x_{u1}$. For $\alpha_{a1} < \alpha < \alpha_{d1}$, J_3^+ and J_3^- are invariant with respect to F^3 and F'^3, respectively, since

$$f \circ f' \circ f : J_3^+ \to J_3^-, \quad f' \circ f \circ f' : J_3^- \to J_3^+ \qquad (9)$$

are satisfied.

That is, the following situations are satisfied.

1. At $\alpha = \alpha_{a1}$, the composite map F^3 is tangent to the Line-B$^+$ (A in Fig. 8) and a tangent bifurcation occurs.
2. At $\alpha = \alpha_{w1}$, $F^3(Q) = Q$ is satisfied (B in Fig. 8). That is, Q becomes a superstable fixed point. If α increases beyond α_{w1}, three extrema appear near Q on F^3 (C in Fig. 8).
3. At $\alpha = \alpha_{d1}$, $F^3(Q)$ is equal to x_{u1} (F in Fig. 8) and an interior crisis occurs.

Therefore, 3-periodic window appears for $\alpha_{a1} < \alpha < \alpha_{d1}$.

Especially, for $\alpha_{b2} < \alpha < \alpha_{r2}$ two invariant intervals J_{3a}^+ and J_{3b}^+ appear in J_3^+ as shown in Fig. 9. In the case these intervals exist, any point in J_{3a}^+ are not transformed into J_{3b}^+ and also any point in J_{3b}^+ are not transformed into J_{3a}^+. Therefore, in this parameter range, two distinct orbits exist; one passes through J_{3a}^+ and J_{3b}^- alternately, and the other passes through J_{3b}^+ and J_{3a}^- alternately.

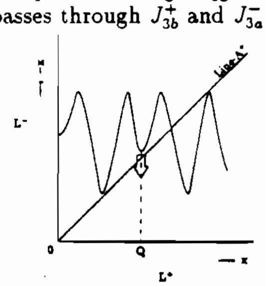


Fig. 6 $f \circ f' \circ f$.

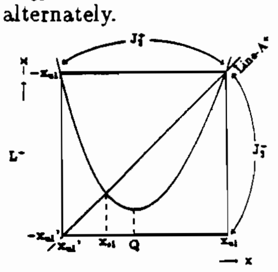


Fig. 7 The neighborhood of Q on $f \circ f' \circ f$.

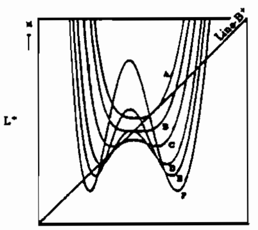


Fig. 8 The neighborhood of Q on F^3.

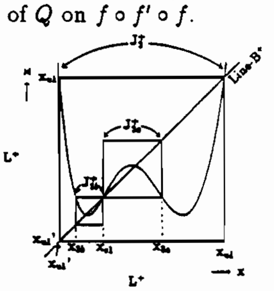


Fig. 9 Two invariant intervals J_{3a}^+ and J_{3b}^+.

3.2 THE WINDOW OF TYPE 2

Next, we discuss the mechanism of the window of Type 2.

Consider the composite maps $f' \circ f \circ f' \circ f$ and $f \circ f' \circ f \circ f'$, that is, F^2 and F'^2. As α increases, $F^2(Q)$ descends down. As far as we carry out the computer simulations, the following situations exist.

[Situation 2]
1. At $\alpha = \alpha_{a2}$, F^2 is tangent to the Line-B$^+$ and a tangent bifurcation occurs. F^2 has two 2-periodic points near Q for $\alpha_{a2} < \alpha < \alpha_{d2}$. At least for $\alpha_a < \alpha < \alpha_w$ one is stable 2-periodic point and the other is unstable 2-periodic point. Let the former be x_{s2} and also the latter be x_{u2}.
2. At $\alpha = \alpha_{w2}$, Q is equal to x_{s1} (superstable).
3. At $\alpha = \alpha_{d2}$, $F^4(Q)$ is equal to x_{u2} and an interior crisis occurs.
4. For $\alpha_{a2} < \alpha < \alpha_{d2}$, invariant intervals $J_4^+ = [x'_{u2}, x_{u2}] \subset J^+$ and $J_4^- = [-x'_{u2}, -x_{u2}] \subset J^-$ can be defined in $J^\pm$, where $F^2(x'_{u2}) = x_{u2}$.

Therefore, 2-periodic window appears for $\alpha_{a2} < \alpha < \alpha_{d2}$.

For the purpose of simple explanation of this mechanism, the maps f and f' at $\alpha = \alpha_{w2}$ are shown with being overlapped in Fig. 10. We consider the orbit starting from Q in J^+. The point Q is transformed into a point in J^- by f. Thereafter, the point in J^- is transformed into a point in J^+ by f'. Figure 10 clearly shows that the orbit starting from Q make the periodic attractor which returns to Q through four periodic points in J^- and J^+ alternately. Note that the positions of the points in J^+ included in the periodic attractor are different from that in J^-. Namely, in this case asymmetric attractor appears in the circuit.

4 SCALING MECHANISMS

As mentioned in the previous section, the composite map $f \circ f' \circ f$ makes the invariant interval J_3^+ near Q for $\alpha_{a1} < \alpha < \alpha_{d1}$. We define two maps T_α and T'_α as follows.

$$\begin{aligned} T_\alpha &= \quad f \circ f' \circ f: \quad J_3^+ \to J_3^-, \\ T'_\alpha &= \quad f' \circ f \circ f': \quad J_3^- \to J_3^+. \end{aligned} \qquad (10)$$

Moreover, we rewrite $J_3^+ = [x'_{u1}, x_{u1}]$ as $I^+ = [l, m]$ and also $J_3^- = [-x'_{u1}, -x_{u1}]$ as $I^- = [-l, -m]$. In that follows, we shall consider about T_α and T'_α for only $\alpha_{a1} < \alpha < \alpha_{d1}$.

[Theorem 1]
If T_α satisfies the following (**Condition**), there exist the windows which appear in the following order.

$$\begin{array}{llllllll} \text{Type 1}: & 3\times 3 & 3\times 5 & 3\times 7 & \cdots & 3\times n & \cdots \\ & \searrow\ \nearrow & \searrow\ \nearrow & \searrow\ \nearrow & \cdots \nearrow & \searrow\ \nearrow & \\ \text{Type 2}: & 3\times 2 & 3\times 3 & 3\times 4 & \cdots & 3\times \frac{n}{2} & \cdots \end{array}$$

$$(n = 3, 4, 5, \cdots) \qquad (11)$$

where the numbers represent periods of the windows when it appears.

(Condition)
1. The map T_α satisfies the Schwarzian condition [4].
2. The map T_α has a unique extremum Q in I^+.
3. As α increases continuously, the map T_α changes continuously from $T_{\alpha_{a1}}$ to $T_{\alpha_{d1}}$ where $T_{\alpha_{a1}}(x) = -x$ is satisfied for only $x = m$ and $T_{\alpha_{d1}}(x) = -l$ is satisfied for only $x = Q$ as shown in Fig. 11.

Since T'_α has the same form as T_α, if T_α satisfies the above conditions, T'_α also satisfies similar conditions.

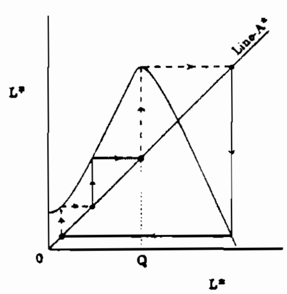


Fig. 10 f.

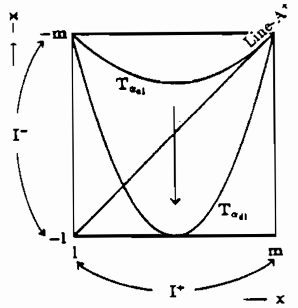


Fig. 11 T_α.

[**Proof**]

At first, we consider the composite map $T'_\alpha \circ T_\alpha$. Here we define the interval $I_2^+ = [l_2, m_2]$ where $T'_\alpha \circ T_\alpha(l_2) = T'_\alpha \circ T_\alpha(m_2) = l_2$.

As α increases, $T'_\alpha \circ T_\alpha(Q)$ changes at least from m_2 to m continuously, because $T_{\alpha_{d1}}(Q) = -l$ is satisfied and $T'_\alpha(-l)$ always is equal to m.

Moreover, we consider about the form of $T_\alpha \circ T'_\alpha \circ T_\alpha$ in I_2^+. In order to investigate the form of $T_\alpha \circ T'_\alpha \circ T_\alpha$, we examine the derivative of $T_\alpha \circ T'_\alpha \circ T_\alpha$. From the form of T_α and $T'_\alpha \circ T_\alpha$, following equation is satisfied.

$$D\{T_\alpha \circ T'_\alpha \circ T_\alpha(x)\} \left\{ \begin{array}{l} < 0 \cdots x \subset [l_2, e_1] \cup [Q, e_2] \\ > 0 \cdots x \subset [e_1, Q] \cup [e_2, m_2] \end{array} \right. \quad (12)$$

where $T'_\alpha \circ T_\alpha(e_1) = T'_\alpha \circ T_\alpha(e_2) = Q$ $(l_2 < e_1 < Q < e_2 < m_2)$. Therefore, the composite map $T_\alpha \circ T'_\alpha \circ T_\alpha$ has three extrema in I_2^+.

As α increases, $T_\alpha \circ T'_\alpha \circ T_\alpha(Q)$ changes at least from $-l_2$ to $-m$, because $T'_{\alpha_{d1}} \circ T_{\alpha_{d1}}(Q) = m$ is satisfied by the above discussion and $T_\alpha(m)$ always is equal to $-m$. Namely, as α increases, the neighborhood of Q ascends and intersects the Line-A$^+$ at some parameter. If Condition 1 (Schwarzian condition) is satisfied, the number of intersections of $T_\alpha \circ T'_\alpha \circ T_\alpha$ and the Line-A$^+$ in I_2^+ is two. Therefore, there exists a parameter range $\alpha_{a3} < \alpha < \alpha_{d3}$ where the similar situations to the **Situation 1** occur in the neighborhood of Q and a window of Type 1 appears. The period of this window is three with respect to T_α, therefore, the period is 3×3 with respect to the original map F.

We define the interval $I_3^+ = [l_3, m_3]$ where $T_\alpha \circ T'_\alpha \circ T_\alpha(l_3) = T_\alpha \circ T'_\alpha \circ T_\alpha(m_3) = -l_3$.

Next, we consider the composite map $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha$ in I_3^+. In order to investigate the form of $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha$, we examine the derivative of $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha$. From the form of T'_α and $T_\alpha \circ T'_\alpha \circ T_\alpha$, the following equation is satisfied.

$$D\{T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha(x)\} \left\{ \begin{array}{l} < 0 \cdots x \subset [l_3, e_3] \cup [Q, e_4] \\ > 0 \cdots x \subset [e_3, Q] \cup [e_4, m_3] \end{array} \right. \quad (13)$$

where $T_\alpha \circ T'_\alpha \circ T_\alpha(e_3) = T_\alpha \circ T'_\alpha \circ T_\alpha(e_4) = Q$ $(l_3 < e_3 < Q < e_4 < m_3)$. Therefore, the composite map $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha$ has three extremum in I_3^+.

As α increases, $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha(Q)$ changes at least from l_2 to m, because $T_{\alpha_{d1}} \circ T'_{\alpha_{d1}} \circ T_{\alpha_{d1}}(Q) = -m$ is satisfied by the above discussion and $T'_\alpha(-m)$ always is equal to m. Namely, as α increases, the neighborhood of Q ascends and intersects the Line-B$^+$ at some parameter. Therefore, there exists a parameter range $\alpha_{a4} < \alpha < \alpha_{d4}$ where the similar situations to the **Situation 2** occur in the neighborhood of Q and a window of Type 2 appears. This window appears after disappearance of the above-mentioned $9(= 3 \times 3)$-periodic window of Type 1, that is, α_{a4} must be larger than α_{d3}, because $T'_{\alpha_{d3}} \circ T_{\alpha_{d3}} \circ T'_{\alpha_{d3}} \circ T_{\alpha_{d3}}(Q)$ is smaller than l_2. The period of this window is two with respect to T_α, therefore, the period is 3×2 with respect to the original map F.

We define the interval $I_4^+ = [l_4, m_4]$ where $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha(l_4) = T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha(m_4) = l_4$.

The form of $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha$ in I_4^+ resembles $T'_\alpha \circ T_\alpha$ in I_2^+. Therefore, the similar situation to those of $T'_\alpha \circ T_\alpha$ in I_2^+ could be considered for $T'_\alpha \circ T_\alpha \circ T'_\alpha \circ T_\alpha$ in I_4^+ and we can repeat the similar discussion. Then we can say $15(= 3 \times 5)$-periodic window of Type 1 appears. In this way, the above discussions can be repeated infinitely. As a result, we get the above [**Theorem 1**]. Q.E.D.

Finally, experimental results are shown in Fig. 12. The windows of Type 1 and Type 2 are shown in Figs. 12(b) and (d), respectively.

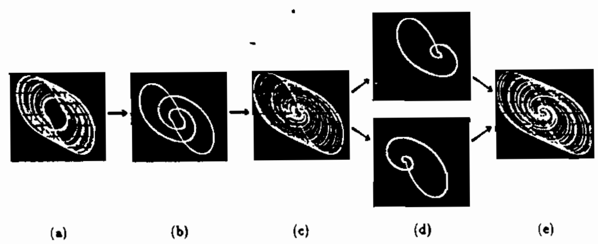


Fig. 12 Experimental results.

5 CONCLUSION

In this paper, the two types of windows generated in a symmetric circuit have been investigated. By using a degeneration technique, we have derived Poincaré map strictly as a one-dimensional map. We make clear the mechanisms of these two types of windows and prove rigorously that the two types of windows appear alternately and infinitely inside some windows with some reasonable assumptions by utilizing a certain scaling structure on the Poincaré map.

The generation of the windows discussed in this paper would be universal in symmetric chaos generating systems, because this circuit is extremely simple.

REFERENCES

[1] J. Testa and G. A. Held, Phys. Rev. A, vol. 28, no. 5, p. 3085, Nov. 1983.

[2] N. Inaba, T. Saito and S. Mori, Trans. IEICE, vol. E-70, p. 744, Aug. 1987.

[3] T. Saito, Trans. IEICE Japan, vol. J71-A, p. 1275, June 1988.

[4] P. Collet and J.-P. Eckmann, Iterated Maps on the Interval as Dynamical Systems, Birkhäuser, Boston, 1980.