STABILITY PAPERS

SIMULTANEOUS OSCILLATIONS IN OSCILLATORS

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SUMMARY - An oscillator with two degrees of freedom can, under certain conditions, oscillate simultaneously at two different frequencies. The ratio of the two frequencies may be rational (synchronous oscillations) or irrational (asynchronous oscillations). The conditions necessary for simultaneous oscillations are discussed for these two cases. The problem is essentially nonlinear and cannot be solved by the methods of linear analysis. The method of equivalent linearization, developed by Kryloff and Bogoliuboff, is shown to give a good correlation with experimental results.

INTRODUCTION

The oscillator shown in Fig. 1, where NL is a negative nonlinear resistance may oscillate either at a single frequency \( \omega_1 \) or \( \omega_2 \), or simultaneously at both these frequencies. Simultaneous oscillations in circuits with several degrees of freedom form one of the most interesting subjects of the theory of oscillations. In the last three years a number of papers dealing with this problem have been published (Refs. 2,3,4,5) and it has been proven theoretically and experimentally that simultaneous oscillations are indeed possible.

No attempt is made in the following to give a rigorously complete theory. It is hoped that this lack of precision will be more than offset by a greater simplicity of the discussion. The reader who prefers a more complete analysis is referred to papers by L. Skinner (Ref. 1) and R. Fontana (Ref. 2).

THE CIRCUIT MODEL

It is assumed that there is no inductive coupling between the two resonant circuits of Fig. 1. It is furthermore assumed that for simultaneous oscillations the voltages \( V_1 \) and \( V_2 \) are approximately:

\[
\begin{align*}
V_1 & \approx V_1 \cos(\omega_1 t + \phi_1) \\
V_2 & \approx V_2 \cos(\omega_2 t + \phi_2)
\end{align*}
\]

where \( \frac{\omega_1}{\omega_2} \) may be rational or irrational.

The current \( i \) passing through the active, nonlinear element and the voltage \( v \) across it are not related by a linear equation. The nonlinear element may be the gap of a reflex klystron, the interaction space of a magnetron, a point contact transistor, etc.

One method for calculating the circuit of Fig. 1 is to find the corresponding differential equation and to solve by an approximation method (Poincare, van der Pol and others) (Ref. 4). These methods can be applied successfully since a very good first approximation from physical considerations is known (Eq. 1). The methods are, however, somewhat questionable from a mathematical point of view since convergence of the higher approximations cannot be assured. For this paper a somewhat different method has been selected, i.e., the method of equivalent linearization developed by Kryloff and Bogoliuboff (Ref. 4). The results found by both methods are identical and a very good correlation with experimental results has been found.

Energy is stored in the two resonant circuits of Fig. 1. This energy changes continually and rapidly between the electromagnetic and electrostatic form. The total will, however, remain almost constant:
PROFESSIONAL GROUP ON CIRCUIT THEORY

at frequency $\omega_1$: $E_1 = \frac{V_1^2}{2} C_1$  \hspace{1cm} (2)

at frequency $\omega_2$: $E_2 = \frac{V_2^2}{2} C_2$

where $E_1$ and $E_2$ are the total energies stored in the two resonant circuits.

A small part of this energy is dissipated in the conductances $G_1$ and $G_2$ and an equally small amount is fed into the circuits by the current $i$ passing through the circuits so that approximately:

\[
\frac{dE_1}{dt} = -G_1 \frac{V_1^2}{2} - \lim_{T \to \infty} \frac{1}{T} \int_0^T i V_1 \cos(\omega_1 t + \phi_1) \, dt \hspace{1cm} (3)
\]

\[
\frac{dE_2}{dt} = -G_2 \frac{V_2^2}{2} - \lim_{T \to \infty} \frac{1}{T} \int_0^T i V_2 \cos(\omega_2 t + \phi_2) \, dt \hspace{1cm} (4)
\]

These equations show that only the components of $i$ at the frequencies $\omega_1$ and $\omega_2$ need be considered.

If a voltage $v = V_1 \cos(\omega_1 t + \phi_1) + V_2 \cos(\omega_2 t + \phi_2)$ is applied to the nonlinear element $NL$, then a current $i$ will flow:

\[
i = I_{11} \cos(\omega_1 t + \phi_1) + I_{12} \sin(\omega_1 t + \phi_1)
+ I_{21} \cos(\omega_2 t + \phi_2) + I_{22} \sin(\omega_2 t + \phi_2)
+ \text{terms at frequencies other than } \omega_1 \text{ or } \omega_2. \hspace{1cm} (5)
\]

For the circuit consisting of $L_1$, $G_1$, $C_1$, only the terms at frequency $\omega_1$ are important. They are:

\[
i = I_{11} \cos(\omega_1 t + \phi_1) + I_{12} \sin(\omega_1 t + \phi_1)
\]

The same current would flow through this resonant circuit if the nonlinear element and the second resonant circuit were replaced by the “equivalent” impedance represented by $G_{1e}$ and $C_{1e}$ of Fig. 2:

\[
G_{1e} = \frac{I_{11}}{V_1}, \quad C_{1e} = \frac{I_{12}}{\omega_1 V_1}, \quad I_{12} = \frac{I_{12}}{\omega_1 V_1}
\]

A similar procedure leads to the circuit 2b with

\[
G_{2e} = \frac{I_{11}}{V_2}, \quad C_{2e} = \frac{I_{12}}{\omega_2 V_2}, \quad I_{12} = \frac{I_{12}}{\omega_2 V_2}
\]

Further considerations will be based on the two circuits of Fig. 2.

SYNCHRONOUS AND ASYNCHRONOUS SIMULTANEOUS OSCILLATIONS

In the following, a distinction is made between “synchronous” and “asynchronous” simultaneous oscillations. For “synchronous” oscillations a relation holds between the two frequencies $\omega_1$ and $\omega_2$ where $p$ and $q$ are “small” integers. (For $|p + q|$ large, the synchronization effect becomes so small that it can no longer counteract the continuous disturbance due to line voltage fluctuation, etc.)

\[
p \omega_1 = q \omega_2 \hspace{1cm} (7)
\]

The oscillations can be divided into two groups: in one, strong synchronization exists; in the other, no rational frequency-ratio is preserved over an appreciable period of time. Mathematically, it is assumed that in this second region $\frac{\omega_1}{\omega_2}$ is constant but irrational. This does not result in any appreciable mistakes since the synchronization effects are small in the region, by assuming $\frac{\omega_1}{\omega_2}$ constant but irrational, they are neglected entirely.

The expression

\[
\phi = p \phi_1 - q \phi_2 = p \phi_1 - q \phi_2 \hspace{1cm} (8)
\]

is defined as the phase angle between the voltages $V_1$ and $V_2$. If $p$ and $q$ are integers, then the value of $\phi$ is determined uniquely under a shift of the time axis. However, for either $p$ or $q$ irrational, it can be adjusted arbitrarily close to any value by a suitable shift of the time axis. Since this shift is a purely mathematical operation and cannot have any influence on the oscillator, it is clear that the equivalent impedances for asynchronous oscillations cannot be functions of $\phi$. For synchronous oscillations, however, the equivalent impedances are functions of the phase angle, and, of course, of the amplitudes $V_1$ and $V_2$. 
THE EQUIVALENT IMPEDANCES

The "equivalent linearization" can be applied to an oscillator if the equivalent impedances can be determined either experimentally or analytically. The analytical method, used in this paper, can be applied when the current-voltage relationship is expressible as a rapidly converging power series:

\[ i = a_1 v + a_2 v^2 + a_3 v^3 + \ldots + a_n v^n + \ldots \]  
(9)

The equivalent impedances can then be calculated by determining the components \( I_{11}, I_{12}, I_{13}, I_{22} \) of the current \( i \) (Eq. 5). This is best done by expanding (9) into a double Fourier series using

\[ v = V_1 \cos(\omega_1 t + \phi_1) + V_2 \cos(\omega_2 t + \phi_2) \]

The equivalent impedances are then found from (6). Only the results of the calculation will be presented here:

For Asynchronous Oscillations,

\[ G_{1e} = a_1 + \frac{a_2}{2} (V_1^2 + 2V_2^2) + \frac{a_3}{3} (V_1^4 + 6V_1^2V_2^2 + 3V_2^4) + \ldots \]
\[ G_{2e} = a_1 + \frac{a_2}{2} (V_1^2 + 2V_2^2) + \frac{a_3}{3} (V_1^4 + 6V_1^2V_2^2 + 3V_2^4) + \ldots \]  
(10)

For Synchronous Oscillations, \((p\omega_1 = q\omega_2)\), and the general expressions for the equivalent conductances are extremely complicated, and only four terms of the series (9) will be considered:

\[ i = a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 \lambda v \]  
(11)

where \( \lambda = p + q - 1 \). The term \( a_4 v^4 \lambda v \) is the lowest term of (9) contributing to \( G_{1e}, G_{2e} \). In most cases, replacing (9) with (11) will lead to good results. The values for \( G_{1e}, G_{2e} \), and \( C_{1e} \) and \( C_{2e} \) are then:

\[ G_{1e} = a_1 + \frac{a_2}{2} (V_1^2 + 2V_2^2) + a_4 p_1 V_1 p_2 V_2 q \cos \Phi \]
\[ G_{2e} = a_1 + \frac{a_2}{2} (V_1^2 + 2V_2^2) + a_4 q_1 V_1 q_2 V_2 q^{-1} \cos \Phi \]  
(12)

\[ C_{1e} = -\frac{a_4 p_1}{\omega_1} V_1 p_2 V_2 q \sin \Phi \]
\[ C_{2e} = -\frac{a_4 q_1}{\omega_2} V_1 q_2 V_2 q^{-1} \sin \Phi \]

where

\[ a_{pq} = -\frac{q\lambda}{2^{\lambda-1}} \left( \frac{q}{p} \right) \]

The following relations hold generally, even if all the terms of (9) are considered:

\[ C_{2e} = -C_{1e} \frac{V_1^2}{V_2^2} \omega_2^2 \]  
(13)

and

\[ \frac{\partial C_{1e}}{\partial V_1} = \frac{\partial C_{2e}}{\partial V_2} \]  
(14)

The synchronization effect is due to \( C_{1e} \) and \( C_{2e} \). For large values of \( \lambda \), the equivalent capacitances \( C_{1e} \) and \( C_{2e} \) are very small and will not be able to lock the frequencies \( \omega_1 \) and \( \omega_2 \) together in the presence of line-fluctuations and noise.

NONSTEADY-STATE OSCILLATIONS

In the following, a distinction is made between steady-state oscillations \((V_1, V_2, \Phi \text{ constant})\) and non-steady-state or "transient" oscillations \((V_1, V_2, \Phi \text{ changing})\). Transient and steady-state oscillations can be calculated from the circuits of Fig. 2.

The conductances of Fig. 2 will damp the oscillations and change the amplitudes. The capacitances \( C_{1e} \) and \( C_{2e} \) will change the frequencies \( \omega_1 \) and \( \omega_2 \) and cause a shift of the phase angle \( \Phi \).

The corresponding differential equations are for asynchronous oscillations:

\[ \frac{dV_1}{dt} = -\frac{V_1}{2C_1} \left[ C_1 + C_{1e} \right] \]  
(15)

\[ \frac{dV_2}{dt} = -\frac{V_2}{2C_2} \left[ C_2 + C_{2e} \right] \]

and for synchronous oscillations:

\[ \frac{dV_1}{dt} = -\frac{V_1}{2C_1} \left[ C_1 + C_{1e} \right] \]  
(16)

\[ \frac{dV_2}{dt} = -\frac{V_2}{2C_2} \left[ C_2 + C_{2e} \right] \]

where

\[ \Delta \omega = \frac{p}{\sqrt{L_1 C_1}} - \frac{q}{\sqrt{L_2 C_2}} \quad \text{and} \quad C_e = C_{1e} - \frac{C_1}{C_2} \]

It is possible to eliminate the time from (15) by division:

\[ \frac{dV_1}{C_1 V_1 - \frac{C_{1e}}{C_2 V_2}} = \frac{dV_2}{C_1 V_1 + \frac{C_{1e}}{C_2 V_2}} \]  
(17)

This differential equation can be solved graphically by the method of isoclines in a plane with \( V_1 \) and \( V_2 \) as coordinates. In Figs. 3 and 4 the trajectories corresponding to (17) have been plotted for a third- and a fifth-degree polynomial. A point in this plane corresponds to an oscillation with the amplitudes \( V_1 \) and \( V_2 \). For non-steady state oscillations, this point will move along a trajectory in the direction indicated by the arrow.

A similar representation in a space with \( V_1, V_2, \Phi \) as coordinates is possible for synchronous oscillations.
The study of the transient oscillations is often important. For example, in a system with several possible steady-state oscillations, the transient study will predict which one of these will be reached from given initial conditions. It will also determine the manner in which oscillations will build up and will describe the behavior of the system if a circuit parameter is changed abruptly.

A steady-state oscillation is stable if the oscillator will resume its original amplitude after a small disturbance. For example, the steady-state oscillations corresponding to point A of Fig. 4 are stable, but not those corresponding to point B.

In order to calculate the behavior of the system in the neighborhood of the steady-state oscillations, it is necessary to expand (15) and (16) around the amplitudes $V_{10}, V_{20}$ and the phase angle $\Phi_0$ corresponding to these steady-state oscillations. Assume small perturbations,

$$\begin{align*}
\Delta V_1 &= V_1 - V_{10} \\
\Delta V_2 &= V_2 - V_{20} \\
\Delta \Phi &= \Phi - \Phi_0
\end{align*}$$

The first terms of the expansion of (15) are then,

$$\begin{align*}
\frac{d\Delta V_1}{dt} &= -\frac{V_{10}}{2C_1} \left[ \frac{\partial G_{1e}}{\partial V_1} \Delta V_1 + \frac{\partial G_{1e}}{\partial V_2} \Delta V_2 \right] \\
\frac{d\Delta V_2}{dt} &= -\frac{V_{20}}{2C_2} \left[ \frac{\partial G_{2e}}{\partial V_1} \Delta V_1 + \frac{\partial G_{2e}}{\partial V_2} \Delta V_2 \right]
\end{align*}$$

and if $\Delta V_1, \Delta V_2$ are small, the higher-order terms of the expansion can be neglected. These simultaneous linear differential equations can be solved by standard methods. The variables $\Delta V_1$ and $\Delta V_2$ will approach zero from any initial value in the neighborhood of the steady-state oscillations if both roots $\mu_1$ and $\mu_2$ of Eq. (22) have negative real parts.
Similarly, steady-state synchronous simultaneous oscillations are stable if all three roots of (23) have negative real parts.

Both roots of Eq. (22) have negative real parts if:

\[
\begin{align*}
&\frac{V_1}{2C_1} \frac{\partial G_{1e}}{\partial V_1} + \mu \frac{V_1}{2C_1} \frac{\partial G_{1e}}{\partial V_1} > 0 \\
&\frac{V_2}{2C_2} \frac{\partial G_{2e}}{\partial V_1} + \mu \frac{V_2}{2C_2} \frac{\partial G_{2e}}{\partial V_1} > 0 \\
&\frac{V_1}{2C_1} \frac{\partial G_{1e}}{\partial V_2} + \mu \frac{V_1}{2C_1} \frac{\partial G_{1e}}{\partial V_2} > 0
\end{align*}
\]

Since \( G_{1e} \) and \( G_{2e} \) are known functions of \( V_1 \) and \( V_2 \), it is possible for a specific oscillator to determine the regions of possible steady-state oscillations upon a plane moving \( V_1 \) and \( V_2 \) as coordinates.

If the current-voltage characteristic can be expressed completely by a polynomial:

\[
i = a_1 v + a_2 v^2 + a_3 v^3
\]

then asynchronous simultaneous oscillations are not possible, as illustrated by Fig. 3. However, for the polynomial \( i + a_4 v + a_5 v^2 + a_6 v^3 + a_7 v^4 \), simultaneous oscillations are possible provided \( a_4 < 0, a_6 > 0 \) and the point in the \( V_1-V_2 \) plane corresponding to the steady-state oscillations falls inside the ellipse showing in Fig. 5 (ref. 5) and represented by

\[
6x^2 + 6y^2 + 8xy - 8(x+y) + 3 = 0,
\]

where

\[
x = -\frac{5a_4}{3} V_1^2 \quad \text{and} \quad y = -\frac{5a_6}{3} V_2^2.
\]

It should be noted from Fig. 4 that for this case the asynchronous simultaneous oscillations are not self-starting.

For synchronous simultaneous oscillations, Eq. (23) will determine the regions of stability. It can be written as:

\[
\mu^3 + p_1 \mu^2 + p_2 \mu + p_3 = 0
\]

All the roots of this equation have negative real parts if

\[
p_1 > 0, p_3 > 0, p_1 p_3 - p_2 > 0,
\]

by the Routh-Hurwitz Criteria.
are also zero. Eq. (23) therefore reduces to

\[
\begin{bmatrix}
V_1 \frac{\partial C_{1e}}{\partial V_1} + \mu & \frac{V_1 \partial G_{1e}}{2C_1 \partial V_2} & 0 \\
\frac{V_2 \partial G_{2e}}{2C_2 \partial V_1} & \frac{V_2 \partial G_{2e}}{2C_2 \partial V_2} + \mu & 0 \\
0 & 0 & \frac{p \omega_1 \partial C_e}{2C_1 \partial \Phi} + \mu
\end{bmatrix} = 0 \tag{28}
\]

One of the three roots of this equation is:

\[
1 = -\frac{p \omega_1 \partial C_e}{2C_1 \partial \Phi}
\]

The other two are determined by

\[
\begin{bmatrix}
\frac{V_1 \partial G_{1e}}{2C_1 \partial V_1} + \mu & \frac{V_1 \partial G_{1e}}{2C_1 \partial V_2} \\
\frac{V_2 \partial G_{2e}}{2C_2 \partial V_1} & \frac{V_2 \partial G_{2e}}{2C_2 \partial V_2} + \mu
\end{bmatrix} = 0 \tag{29}
\]

This leads to the following conditions necessary and sufficient for stability:

\[
\frac{V_1 \partial G_{1e}}{2C_1 \partial V_1} + \frac{V_2 \partial G_{2e}}{2C_2 \partial V_2} > 0
\]

\[
\frac{\partial C_{1e}}{\partial V_1} + \frac{\partial C_{2e}}{\partial V_2} > 0
\]

\[
\frac{\partial C_e}{\partial \Phi} > 0
\]

The last of these inequalities is satisfied if \(a_{pq} \cos \Phi < 0\), or

\[
a \lambda > 0 \quad \text{for} \quad \Phi = \pi,
\]

\[
a \lambda < 0 \quad \text{for} \quad \Phi = 0. \tag{31}\]

Of particular interest for physical applications is:

\[
\omega_2 = n \omega_1
\]

A simple physical picture of this case is given. The main oscillation occurs at the frequency \(\omega_1\). Due to the nonlinearity, the current \(i\) will also contain components at a frequency \(n \omega_1\). The current now passes through the second resonant circuit, which is tuned approximately to the frequency \(n \omega_1\) and will excite it. This picture is somewhat oversimplified since the resonant circuit turned to \(n \omega_1\) will, in turn, influence that tuned to \(\omega_1\).

The special case \(\omega_2 = 3 \omega_1\) has been calculated. For \(\Delta \omega = 0\), stable oscillations may exist for \(\Phi = \pi\) and \(V_1/V_2 < 0.50\). Trajectories similar to those of Fig. 4 are shown in Fig. 6 for \(\Delta \omega = 0\).

**Fig. 6 - Transient behavior, synchronous simultaneous oscillations.**

**THE APPLICATION TO MICROWAVE OSCILLATORS**

Due to the many modes of their cavities, microwave oscillators are inherently capable of simultaneous oscillations. In many instances these oscillations are parasitic and result in undesirable disturbances in the frequency-power characteristics (Ref. 3). Since synchronous simultaneous oscillations are not self-starting and are rather difficult to obtain, it is reasonable to assume that simultaneous oscillations observed in microwave generators are synchronous.

However, synchronous oscillations may be used to generate power at very high frequencies which are exact multiples of a lower frequency of oscillation. For example, if the modes of an oscillator are tuned so that their frequencies are related by \(\omega_2 = n \omega_1\), then power can be extracted at the higher of these two frequencies while the oscillation at the lower frequency is possibly synchronized by an outside source. This procedure, which is called "harmonic loading," has been applied to magnetrons.
CONCLUSIONS

It has been shown that an oscillator with two degrees of freedom may oscillate simultaneously at two different frequencies. The ratio of these two frequencies may either be rational (synchronous simultaneous oscillations) or irrational (asynchronous simultaneous oscillations). Asynchronous oscillations are hard to obtain experimentally, but are easy to treat analytically because the relative phase does not enter the calculations. Synchronous oscillations are easy to obtain experimentally but rather more difficult to treat analytically.

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SYNTHESIS OF TRANSFER FUNCTIONS BY ACTIVE RC NETWORKS*

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SUMMARY — The paper considers the synthesis of arbitrary passive transfer functions using stable active RC networks. The work undertaken is oriented toward discussing general methods of synthesis rather than specific design procedures. The structures used consist of single unidirectional feedback loops. These methods, although by no means providing the most general procedure for the synthesis of active networks, appear to offer a logical step toward this aim.

I. INTRODUCTION

The synthesis of arbitrary transfer functions may in general be accomplished by the use of passive or active RLC networks, or by using active RL or RC networks employing feedback. However, if the important frequency range of the network is confined to low frequencies, the use of inductors becomes prohibitive in practice. Indeed, inductors are in general less attractive than capacitors and resistors, in view of their comparative imperfections.

One is therefore attracted to the last-mentioned alternative, namely, active RC networks with feedback. The purpose of this paper is to investigate methods of synthesizing such circuits by means of simple feedback loops.

A number of authors have already presented details particular to synthesis procedures using active networks. For instance Truxal (Ref. 1) based on principles suggested by E. A. Guillemin offers a simple servo system. W. Kautz (Ref. 2) also considers the synthesis of a transfer function using active RC networks. More recently J. G. Linvill (Ref. 3) presented a method of synthesis using transistors as active elements. Max Matthews and W. W. Seifert have also presented useful methods of realizing transfer functions using relatively simple passive networks (Ref. 4).

Most of the above-mentioned work is directed to the actual circuit design. However, the aim of this article is to present a general discussion on the structural synthesis of transfer functions rather than detailed design procedures. The study of the structures using unidirec-