Multimode Oscillations in Mutually Coupled van der Pol Type Oscillators with Fifth-Power Nonlinear Characteristics

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Abstract—Coupled van der Pol oscillators containing a fifth-order conductance characteristic are analyzed using a method due to Endo and Mori. This structure has been proposed for modeling the myoelectrical activity of the human large intestine since it produces stable conditions of zero, two single-mode frequencies, and also a double-mode condition. The paper first considers the case of oscillators having equal frequencies and it is shown that for a certain range of parameters double modes are possible for two oscillators, which is in contrast to the case of conventional van der Pol dynamics. The theoretical results have been experimentally verified using both analog simulations and electronic circuitry. In digestive tract modeling, the coupled oscillators must often have unequal intrinsic frequencies and this case is also considered and shown to give similar but more complex parameter conditions for the establishment of double modes.

I. INTRODUCTION

The analysis presented here has been prompted by attempts to model the electrical rhythms which can be recorded from the gastrointestinal tract of mammals [1]. The proposed model comprises either a matrix structure of intercoupled nonlinear oscillators for the stomach or a similar ladder structure for the small and large intestines. The nonlinear oscillators have usually been based on van der Pol dynamics although work has been done on coupled systems based on physiologically meaningful Hodgkin-Huxley dynamics [2], [3]. For the human large intestine, two electrical rhythms have been recorded interspersed with periods of electrical silence [4], and to model these conditions the inclusion of a fifth-power term in the van der Pol equation has been investigated [5]. In human large-intestinal electrical recordings there are occasional periods when both the rhythms are present simultaneously. The possibility of this double-mode behavior had been noted from simulation studies [5] and the analysis of it is presented here.

Analysis of mutually coupled nonlinear oscillators for single-mode conditions has been done using the harmonic balance method for a number of different methods of coupling [6]–[9]. The existence of nonresonant simultaneous oscillations, referred to here as double modes, in a nonlinear oscillator with two degrees of freedom has been investigated using harmonic and power balancing methods [10]–[12]. More recently, ladder networks of nonlinear oscillators having equal intrinsic frequencies have
been considered by Scott [13] and Endo and Mori [14]. It has been shown that dual modes can exist in such a ladder structure with open-circuited ends but that they do not occur for the two-oscillator case and that when they exist the component frequencies are not identical to the stable single-mode frequencies.

In Section II the matrix method of Endo and Mori is used on the fifth-power van der Pol dynamic for the case of two inductively coupled oscillators having equal uncoupled frequencies. The stability of single modes including the zero state is demonstrated, followed by a determination of the parameter values for which the two single modes can be excited simultaneously. Analog simulation and electronic hardware verification of these concepts in Section III is followed by an analytical treatment of the case of unequal frequencies in Section IV.

II. OSCILLATORS WITH EQUAL FREQUENCIES

A. Derivation of Mode Equations

A van der Pol oscillator with fifth-power term is of the form

$$\ddot{x} + \xi (b - cx^2 + dx^4)\dot{x} + \omega^2 x = 0. \tag{1}$$

Two of these mutually coupled in the \( x \) to \( \dot{x} \) configuration result in

$$\ddot{x}_1 + \xi (b - cx_1^2 + dx_1^4)\dot{x}_1 + \omega^2 x_1 - \alpha x_1 = 0 \tag{2}$$

$$\ddot{x}_2 + \xi (b - cx_2^2 + dx_2^4)\dot{x}_2 + \omega^2 x_2 - \alpha x_2 = 0 \tag{3}$$

where \( \alpha \) is the coupling factor.

Equations (2) and (3) may be more conveniently rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \omega^2 - \alpha \\ -\alpha \omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\xi \begin{bmatrix} (b - cx_1^2 + dx_1^4) \dot{x}_1 \\ (b - cx_2^2 + dx_2^4) \dot{x}_2 \end{bmatrix}. \tag{4}$$

Provided \( \xi \) is small (< 0.1) so that the oscillation of each of the oscillators can be regarded as almost purely sinusoidal, (4) can be transformed by a unit orthogonal transformation into its separate component modes.

Let \( \tilde{x} = p\vec{x} \) so that \( p \) is the unit orthogonal matrix then we have

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \omega^2 - \alpha \\ -\alpha \omega^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -\xi \begin{bmatrix} b \frac{dx_1}{dt} - \frac{1}{3} c \frac{d(x_1^3)}{dt} + \frac{1}{5} d \frac{d(x_1^5)}{dt} \\ b \frac{dx_2}{dt} - \frac{1}{3} c \frac{d(x_2^3)}{dt} + \frac{1}{5} d \frac{d(x_2^5)}{dt} \end{bmatrix}. \tag{5}$$

The eigenvalues and the eigenvectors in (5) are easily determined to yield

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} \omega^2 - \alpha & 0 \\ 0 & \omega^2 + \alpha \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -\xi \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} b \frac{dx_1}{dt} - \frac{1}{3} c \frac{d(x_1^3)}{dt} + \frac{1}{5} d \frac{d(x_1^5)}{dt} \\ b \frac{dx_2}{dt} - \frac{1}{3} c \frac{d(x_2^3)}{dt} + \frac{1}{5} d \frac{d(x_2^5)}{dt} \end{bmatrix}. \tag{6}$$

The modes of (6) are given by

$$y_i = A_i \cos (\theta_i t + \phi_i), \quad i = 1, 2 \tag{7}$$

where

$$\theta_1 = \sqrt{\omega^2 - \alpha} \quad \text{and} \quad \theta_2 = \sqrt{\omega^2 + \alpha}. \tag{8}$$

In order to test the stability of the modes in (6) the Kryloff and Bogoliuboff linearization technique is used to linearize the cubic and the fifth-order terms in (6). From (5)

$$x_i^3 = (p_{11}y_1 + p_{12}y_2)^3 = p_{11}^3y_1^3 + 3p_{11}^2y_1^2p_{12}y_2 + 3p_{11}y_1p_{12}^2y_2^2 + p_{12}^3y_2^3 \tag{9}$$

substituting (7) into (8) yields

$$x_i^3 = p_{11}^3A_i^3 + 3p_{11}^2A_i^2A_2 \cos (\theta_1 t + \phi_1) \cos (\theta_2 t + \phi_2) + 3p_{11}p_{12}A_i^2A_2^2 \cos^2 (\theta_1 t + \phi_1) \cos (\theta_2 t + \phi_2) + 3p_{11}^2p_{12}A_iA_2^2 \cos^2 (\theta_2 t + \phi_2) \cos (\theta_1 t + \phi_1) \tag{10}$$

now

$$\cos^2 \theta = \frac{1}{2} (\cos 3\theta + 3 \cos \theta) \tag{11}$$

substituting (10) into (9) yields

$$x_i^3 = \frac{1}{4} p_{11}^3A_i^3 \cos 3\gamma_1 + 3 \cos \gamma_1 \tag{12}$$

$$+ \frac{1}{4} p_{12}^3A_i^3 \cos 3\gamma_2 + 3 \cos \gamma_2 \tag{13}$$

$$+ \frac{3}{2} p_{11}^2p_{12}A_iA_2^2 \cos 2\gamma_1 + 1 \cos \gamma_2 \tag{14}$$

$$+ \frac{3}{2} p_{12}^2p_{11}A_iA_2^2 \cos 2\gamma_2 + 1 \cos \gamma_1 \tag{15}$$

where

$$\gamma = \theta_1 t + \phi.$$
In the postulated solution to (6), harmonic and cross-coupling terms have been neglected since only nonresonant modes are being considered, and hence (11) simplifies to

\[ x_1 = \frac{3}{4} p_{11} A_1^3 \cos \gamma_1 + \frac{3}{4} p_{12} A_1^2 \cos \gamma_2 + \frac{3}{4} p_{11} p_{12} A_1 A_2 \cos \gamma_1 \]

i.e.,

\[ x_1 = \frac{3}{4} p_{11} A_1^3 \gamma_1 + \frac{3}{4} p_{12} A_2 \gamma_2 \]

or

\[ x_1 = \frac{3}{4} p_{11} A_1 y_1 + \frac{3}{4} p_{12} A_1 y_2 \]

Similarly, \( x_2 \) may be written as

\[ x_2 = \frac{3}{4} p_{21} A_2^3 \gamma_1 + \frac{3}{4} p_{22} A_2 \gamma_2 \]

The fifth-order term may also be simplified using an identical method. From (5)

\[ x_1 = p_{11} y_1 + p_{12} y_2 + 5 p_{11} p_{12} y_1 y_2 + 5 p_{11} p_{12} y_1 y_2 + 10 p_{11} p_{12} y_1^2 y_2 + 15 p_{11} p_{12} y_1 y_2. \]  

Substituting (7) and simplifying reduces (15) to

\[ x_1 = \frac{3}{4} p_{11} A_1^4 y_1 + \frac{3}{4} p_{12} A_2^4 y_2 + \frac{15}{8} p_{11} p_{12} A_1^2 A_2 \gamma_1 + \frac{30}{8} p_{11} p_{12} A_1^2 A_2 \gamma_2 + \frac{30}{8} p_{11} p_{12} A_1 A_2 \gamma_1 + \frac{30}{8} p_{11} p_{12} A_1 A_2 \gamma_2. \]  

Substituting (13), (14), (16), (17), into (6) modifies it to

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\end{bmatrix} =
\begin{bmatrix}
\omega^2 - \alpha & 0 \\
0 & \omega^2 + \alpha
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix}
\]

\[ = -\xi \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix} \]  

where

\[ \beta_{11} = \left[ -b + \frac{1}{2} \left( \frac{5}{8} p_{11} A_1^4 + \frac{15}{8} p_{11} p_{12} A_2 A_1^2 \right) \right] \]

\[ + \frac{1}{2} \left[ \frac{5}{8} p_{11} A_1^4 + \frac{15}{8} p_{11} p_{12} A_2 A_1^2 + \frac{30}{8} p_{11} p_{12} A_2 A_1^2 A_2 \right] \]

\[ \beta_{12} = \left[ -b + \frac{1}{2} \left( \frac{5}{8} p_{12} A_2^4 + \frac{15}{8} p_{11} p_{12} A_2 A_2^2 \right) \right] \]

\[ + \frac{1}{2} \left[ \frac{5}{8} p_{12} A_2^4 + \frac{15}{8} p_{11} p_{12} A_2 A_2^2 + \frac{30}{8} p_{11} p_{12} A_2 A_2^2 A_2 \right] \]

\[ \beta_{21} = \left[ -b + \frac{1}{2} \left( \frac{5}{8} p_{21} A_1^4 + \frac{15}{8} p_{21} p_{22} A_2 A_1^2 \right) \right] \]

\[ + \frac{1}{2} \left[ \frac{5}{8} p_{21} A_1^4 + \frac{15}{8} p_{21} p_{22} A_2 A_1^2 + \frac{30}{8} p_{21} p_{22} A_2 A_1^2 A_2 \right] \]

\[ \beta_{22} = \left[ -b + \frac{1}{2} \left( \frac{5}{8} p_{22} A_2^4 + \frac{15}{8} p_{21} p_{22} A_2 A_2^2 \right) \right] \]

Further simplification of (18) and numerical substitution of the \( p_{ij} \) reduces (18) to

\[ \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\end{bmatrix} =
\begin{bmatrix}
\omega^2 - \alpha & 0 \\
0 & \omega^2 + \alpha
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix}
\]

\[ = -\xi \begin{bmatrix}
b - \frac{1}{8} (A_1^2 + 2A_2^2) + \frac{3}{8} (A_1^4 + 3A_1^2 A_2 A_2^2) \\
-b - \frac{1}{8} (A_1^2 + 2A_2^2) + \frac{3}{8} (A_1^4 + 3A_1^2 A_2 A_2^2)
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix} \]  

B. Stability of the Zero, Single, and Double Modes

In order to determine which of the modes are stable, it is necessary to determine first the stationary values of the amplitude. Assuming that the amplitudes and phases vary slowly as functions of time, then by substituting (7) into (19) and averaging, it is found that the nonzero equations are

\[ \frac{d(A_i^2)}{dt} = \frac{d(A_i^2)}{dt} \]

\[ = \xi A_i^2 \left[ -b + \frac{1}{8} (A_1^2 + 2A_2^2) - \frac{3}{8} (A_1^4 + 3A_1^2 A_2 A_2^2) \right] \]

\[ = \xi A_i^2 \left[ -b + \frac{1}{8} (A_2^2 + 2A_1^2) - \frac{3}{8} (A_1^4 + 3A_1^2 A_2 A_2^2) \right] \]  

\[ \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\end{bmatrix} =
\begin{bmatrix}
\omega^2 - \alpha & 0 \\
0 & \omega^2 + \alpha
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix}
\]

It should be noted that the ratio of \( \theta_1 / \theta_2 \) is assumed to be nonrational since resonance phenomena [15] have not been encountered in gastrointestinal data. Stationary values of the amplitude are found by equating (20) to zero.

The mode stability is determined by introducing a small perturbation around the stationary point and determining whether all the eigenvalues of the resultant Jacobian

\[ J = \frac{d}{dA_i^2} \left( \frac{d(A_i^2)}{dt} \right) \]

have negative real parts.

The elements of Jacobian associated with (20) are

\[ J_{11} = \xi \left[ -b + \frac{1}{8} (A_1^2 + A_2^2) \right] - \frac{3}{8} d(A_1^4 + 4A_1^2 A_2 A_2^2) \]

\[ J_{12} = \xi \left[ -b + \frac{1}{8} (A_1^2 + A_2^2) \right] - \frac{3}{8} d(A_1^4 + 4A_1^2 A_2 A_2^2) \]

\[ J_{21} = \xi \left[ -b + \frac{1}{8} (A_1^2 + A_2^2) \right] - \frac{3}{8} d(A_2^4 + 4A_1^2 A_2 A_2^2) \]

\[ J_{22} = \xi \left[ -b + \frac{1}{8} (A_1^2 + A_2^2) \right] - \frac{3}{8} d(A_2^4 + 4A_1^2 A_2 A_2^2) \].

1) Stability of the Zero State: For the zero state to exist \( A_1^2 = A_2^2 = 0 \). Hence from (21) it is apparent that the characteristic roots are negative.
2) Stability of the Single Modes: For the single modes to be stable, either $A_1^2$ or $A_2^2 = 0$. Assume $A_2^2 = 0$, then from (20) stationary value of $A_1^2$ is given by

$$-b + \frac{c}{8} A_1^2 - \frac{d}{32} A_1^4 = 0$$

\[\therefore d A_1^4 - 4c A_1^2 + 32b = 0\]

\[A_1^2 = \frac{2c + 2\sqrt{c^2 - 8bd}}{d}\]  

(22a)

or

$$A_1^2 = \frac{2c - 2\sqrt{c^2 - 8bd}}{d}. (22b)$$

The stability matrix for single mode from (21) is

$$\begin{bmatrix}
-\frac{b}{d} + \frac{c}{32} A_1^2 & \frac{c}{32} A_1^2 - \frac{d}{16} A_1^4 \\
0 & -\frac{b}{d} + \frac{c}{32} A_1^2
\end{bmatrix}.$$  

(23)

From (22) and (23) it is apparent that single modes are stable provided that

$$(3d A_1^4 - 8c A_1^2 + 32b) > 0$$

(24)

where $A_1^2$ is given by either (22a) or (22b). Substituting (22a) into (24) simplifies, after some algebraic manipulations, to

$$c^2 - 8bd + c \sqrt{c^2 - 8bd} > 0.$$  

(25)

From (25) it is apparent that for stable single modes to exist with amplitude given by (22a)

$$c^2 - 8bd > 0.$$  

(26)

If (22b) is substituted into (24) then it can be shown that (24) cannot be satisfied. Hence amplitude given by (22b) represents an unstable condition.

3) Double-Mode Stability: For the double mode to be stable, both $A_1$ and $A_2$ must exist. Setting (20) to zero it may be shown that

$$(A_1^2 - A_2^2) \left[ \frac{d}{16} (A_1^2 + A_2^2) - \frac{c}{8} \right] = 0.$$  

(27)

Hence, either $A_1^2 = A_2^2$ or $A_1^2 + A_2^2 = 2c/d$.

Hence the stationary values for the amplitudes, when $A_1^2 = A_2^2$, are given by

$$-b + \frac{3}{8} c A_1^2 - \frac{d}{16} A_1^4 = 0$$

\[\therefore A_1^2 = \frac{3c + \sqrt{9c^2 - 80bd}}{5d}\]  

(29a)

or

$$A_1^2 = \frac{3c - \sqrt{9c^2 - 80bd}}{5d}.$$  

(29b)

The corresponding stability matrix for the double modes is written from (21) as

$$\begin{bmatrix}
-\frac{b}{2} + \frac{c}{16} d A_1^4 & \frac{A_1^2}{4} - \frac{3}{8} d A_1^4 \\
\frac{A_1^2}{4} - \frac{3}{8} d A_1^4 & -\frac{b}{2} + \frac{c}{16} d A_1^4
\end{bmatrix}$$  

(30)

The characteristic roots of (30) must both be negative for a stable double mode to exist. The conditions that have to be satisfied are

$$-16b + 4c A_1^2 - 3d A_1^4 < 0$$  

(31)

and

$$-16b + 12c A_1^2 - 15d A_1^4 < 0.$$  

(32)

Substituting (29a) into (31) and (32) leads, after some algebraic manipulations, to the inequality

$$53.33bd - 6c^2 < -c \sqrt{36c^2 - 320bd} < 160bd - 6c^2.$$  

(33)

The values of the parameters $b$, $c$, $d$ satisfying (26) and (33) simultaneously are sufficient to allow null, single, and nonresonant double modes to exist. If (29b) is substituted into (32), it is found that the inequality is not satisfied and hence the amplitude given by (29b) is not stable.

By carrying out the algebraic manipulations as indicated in (29)-(32) it can also be shown that $A_1^2 + A_2^2 = 2c/d$ is in fact an unstable condition.

III. Experimental Verification

Two "x to x" coupled (inductively coupled in the equivalent circuit sense) van der Pol oscillators with fifth-order nonlinear terms were simulated on an AD4 analog computer. The range of possible values of the parameters $b, c, d$ that can be chosen is quite large. It was found that from the point of view of amplitude scaling on the analog computer, the values of $b = 0.11, c = 0.7, \omega = 1, a = 0.3$ were quite suitable. The simulations were run at 1000 times the nominal speed. The intrinsic frequency of each of the oscillators was set nominally at 159.2 Hz. The expected mode frequencies from (7) were then 181.5 and 133.2 Hz.

With these parameters, the simulations were run for a range of values of $d$ with amplitude scaling of 4 and 2 at the lower and higher range of $d$, respectively.

From the inequality in (33), the expected range of values of $d$ for single modes to be stable was

$$0 < d < 0.55$$

and the expected range of values of $d$ for double modes to be stable was

$$0.28 < d < 0.50.$$  

The results are summarised in Table I. The symbols Z, S, D represent zero, single, and double mode, respectively.
TABLE I

<table>
<thead>
<tr>
<th>d</th>
<th>Theoretically Predicted Modes</th>
<th>Experimentally Observed Modes</th>
<th>Mode Frequencies</th>
<th>Amplitude Scale Factor</th>
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It is apparent from Table I that the experimental results correlate with the theoretically predicted results except for the very bottom and the top ends of the range where the double mode, if it existed, would be expected to be just stable.

These oscillators are not self-starting and hence the mode of oscillation depends on the initial conditions setup in the oscillators. On the analog computer, the transients were simulated by initial conditions applied to the x integrators of the two oscillators. In order to relate the transient behavior to the eventual steady-state mode, values of initial conditions $x_1$ and $x_2$ in the range of

$$-1 < x_1 < 1$$
$$-1 < x_2 < 1$$

were applied and the various modes observed. A two-dimensional plot of these can be seen in Fig. 1.

It is apparent from Fig. 1 that there is a definite threshold value below which oscillations cannot be observed with the value of threshold being a function of $b$, $c$, and $d$. If the magnitude of the initial condition is below the threshold in one oscillator and above the threshold in the other then double-mode oscillations result. Single-mode oscillations occur when the magnitude of the initial conditions is above the threshold in each of the oscillators. If the initial conditions are in-phase, the resultant oscillation is in-phase and vice versa for antiphase oscillations.

Electronic oscillators based on an approximation to the dynamics of (1) have also been constructed and shown to have the behavior predicted by this analysis. The nonlinear damping term in (1) was approximated by the straight-line characteristic shown in Fig. 2(a) and this nonlinear conductance was constructed using the circuit shown in Fig. 2(b). It was found that when two oscillators of this type were coupled inductively, stable conditions of zero, two single-mode frequencies, and one double mode were obtained. The double mode is shown in Fig. 3 in which it can be seen that for the particular circuit parameters used the ratio of the two component frequencies was nearly 2:1 which is the condition that has been recorded from the human colon [4].
IV. OSCILLATORS WITH UNEQUAL FREQUENCIES

Consider two van der Pol oscillators with fifth-order nonlinearity and different intrinsic frequencies coupled as in (34) below:

\[
\begin{align*}
\ddot{x}_1 + \xi (b - cx_1^2 + dx_1^4) \dot{x}_1 + \omega_1^2 x_1 - \alpha x_2 &= 0 \\
\ddot{x}_2 + \xi (b - cx_2^2 + dx_2^4) \dot{x}_2 + \omega_2^2 x_2 - \alpha x_1 &= 0 \\
\end{align*}
\]

writing concisely as a matrix leads to

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
\omega_1^2 - \alpha & 0 \\
0 & \omega_2^2 - \alpha
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
- \xi
\begin{bmatrix}
(b - cx_1^2 + dx_1^4) \dot{x}_1 \\
(b - cx_2^2 + dx_2^4) \dot{x}_2
\end{bmatrix}.
\]

Applying the transformation \( \bar{X} = \rho \bar{x} \) as in (5) leads to

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}
\begin{bmatrix}
\omega_1^2 - \alpha & 0 \\
0 & \omega_2^2 - \alpha
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
- \xi
\begin{bmatrix}
(b - cx_1^2 + dx_1^4) \dot{x}_1 \\
(b - cx_2^2 + dx_2^4) \dot{x}_2
\end{bmatrix}.
\]

The eigenvalues and eigenvectors from (36) are given by

\[
\lambda_1 = \frac{1}{2}(\omega_1^2 + \omega_2^2 - 2 \gamma) \\
\lambda_2 = \frac{1}{2}(\omega_1^2 + \omega_2^2 + 2 \gamma)
\]

where

\[
2 \gamma = \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4 \alpha^2}
\]

\[
p_{11} = \frac{\alpha^2}{2 \lambda_1^2 + \gamma (\omega_1^2 - \omega_2^2)} - p_{22}
\]

\[
p_{12} = -\frac{2 \gamma^2 + \gamma (\omega_1^2 - \omega_2^2) - \alpha^2}{2 \gamma^2 + \gamma (\omega_1^2 - \omega_2^2)} = p_{21}.
\]

In order to determine the stationary values of amplitude the method of Section II-A gives

\[
\frac{d(A_i^2)}{dt} = -\xi A_i^2 \left( b + c \left( \frac{1}{4} A_i^2 (p_{11}^2 + p_{12}^2) + A_i^2 (p_{11}^2 + p_{12}^2) \right) \right)
\]

\[
- d \left( \frac{1}{4} A_i^2 (p_{11}^2 + p_{12}^2) + \frac{1}{8} A_i^2 A_i^2 (p_{11}^2 + p_{12}^2 + p_{12}^2 p_{21}^2) \right)
\]

\[
+ \frac{1}{8} A_i^2 (p_{11}^2 p_{21}^2 + p_{12}^2 p_{21}^2) \right). \]

Also \(d(A_i^2)/dt\) can be obtained by simply changing \( A_i^2 \) to \( A_i^2 \) to \( A_i^2 \) to \( A_i^2 \) and \( p_{12}^2 \) to \( p_{22}^2 \).

In order to determine stability, the generalized equation equivalent to (21) is

\[
J_{11} = \frac{\partial}{\partial A_i^2} \left( \frac{d(A_i^2)}{dt} \right)
\]

\[
= \xi \left( - b + c \left( \frac{1}{4} A_i^2 (p_{11}^2 + p_{12}^2) + A_i^2 (p_{11}^2 + p_{12}^2) \right) \right)
\]

\[
- d \left( \frac{1}{4} A_i^2 (p_{11}^2 + p_{12}^2) + \frac{1}{8} A_i^2 A_i^2 (p_{11}^2 + p_{12}^2 + p_{12}^2 p_{21}^2) \right)
\]

\[
+ \frac{1}{8} A_i^2 (p_{11}^2 p_{21}^2 + p_{12}^2 p_{21}^2) \right). \]

Also \( J_{21} \) and \( J_{22} \) can be obtained by changing \( A_i^2 \) to \( A_i^2 \) to \( A_i^2 \) and \( p_{12}^2 \) to \( p_{22}^2 \) in (43) and (44), respectively.
Let
\[ e = p_{11}^2 + p_{12}^2 \quad f = p_{11}^2 + p_{12}^2 \]
\[ g = p_{11}^2 + p_{12}^2 \quad h = p_{11}^2 + p_{12}^2 \] and
\[ A_i = A, \quad g = p_i, + p_j, h = p_i^2 + p_j^2 \] (45)
then single-mode stationary values may be determined from (42) as
\[ s_{11} = s_{22} = -b + c \left[ \frac{A_i^2}{2} e + A_i^2 f \right] - d\left[ \frac{3}{8} A_i^2 g + \frac{15}{8} A_i^2 h \right] \]
(52)
\[ s_{12} = s_{21} = \left\{ cA_i^2 f - \frac{3}{8} dhA_i^4 \right\} \] (53)
also, the single-mode stability condition from (43), is
\[ \xi\left( -b + \frac{3}{8} A_i^2 e - d^2 \left[ \frac{1}{8} A_i^2 g \right] \right) < 0. \] (47)
Substituting (46) and (47), after some simplification, leads to
\[ 8b - c^2 e^2 + c e \sqrt{c^2 e^2 - 8bgd} < 0. \] (48)
Hence the following condition must be satisfied for single-mode stationarity
\[ c^2 e^2 - 8bgd > 0. \] (49)
Double-mode stability conditions are obtained from the stability matrix elements \( s_{ij} \) assuming \( A_i = A_2 \)
\[ s_{11} = s_{22} = -b + c \left[ \frac{A_i^2}{2} e + A_i^2 f \right] - d\left[ \frac{3}{8} A_i^2 g + \frac{15}{8} A_i^2 h \right] \]
(52)
Conditions for both the characteristics roots to be negative are given by
\[ -b + \frac{3}{8} eA_i^2 - d\left[ \frac{3}{8} A_i^2 g + \frac{15}{8} A_i^2 h \right] < 0 \] (54)
\[ -b + c \left[ \frac{A_i^2}{2} e + 2A_i^2 f \right] - d\left[ \frac{3}{8} A_i^2 g + \frac{27}{8} A_i^2 h \right] < 0. \] (55)
Equation (57) and (58) must be satisfied simultaneously for stable double modes to exist.

Double-mode stationary values, assuming equal amplitudes, are derived from (42), as
\[ -b + c\left[ \frac{3}{8} A_i^2 e + A_i^2 f \right] - d\left[ \frac{1}{8} A_i^2 g + \frac{3}{8} A_i^2 h \right] = 0 \] (50)
which reduces to
\[ A_i^2 = \frac{c(e+4f)\pm \sqrt{c^2(e+4f)^2 - 32bd(g+9h)}}{4d(g+9h)} \] (51)
Double-mode stationary values, assuming equal amplitudes, are derived from (42), as
\[ -b + c\left[ \frac{3}{8} A_i^2 e + A_i^2 f \right] - d\left[ \frac{1}{8} A_i^2 g + \frac{3}{8} A_i^2 h \right] = 0 \] (50)
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\[ A_i^2 = \frac{c(e+4f)\pm \sqrt{c^2(e+4f)^2 - 32bd(g+9h)}}{4d(g+9h)} \] (51)

V. Conclusions

It has been shown that the Kryloff–Bogoliuboff linearization method applied to coupled oscillator systems by Endo and Mori can be used to investigate the existence of double modes in a hypothesized model of the electrical activity in the human large intestine. The four known physiological conditions of a zero state, two single-mode
rhythms, and a double mode which had been observed in simulation studies have now been analyzed theoretically. Unlike the case of conventional van der Pol dynamics which cannot give double modes for a two-oscillator case, it has been shown that the addition of a fifth-power term and suitable choice of parameters can give a dual mode which comprises components of the single-mode frequencies.

The incorporation of the fifth-power term and the resulting dual mode has been investigated via analog simulation and electronic hardware. In the latter case, the nonlinear conductance term was synthesized using simple components and a 16 oscillator ladder structure is now being investigated as the basis for a hardware model of the human colon. The two-oscillator case considered here appears to be a relevant subunit of a larger model since for the antiphase single mode there is a large energy transfer between the oscillators giving rise to a higher entrainment frequency in the case of inductive coupling and a lower entrainment frequency for capacitive coupling. For ladders of more than two oscillators with small frequency gradient such as in the human colon, the antiphase single mode is such that adjacent pairs of oscillators are in-phase so that under stable conditions with equal amplitude there is no energy transfer between the pairs. This concept has also been noticed in a ladder of four fifth-power oscillators which gave a double mode equivalent to the two-oscillator case with equal waveforms on oscillators 2 and 3 indicating an absence of energy transfer between pairs.

In some parts of the digestive tract, particularly in the duodenum, it is known that although the intact organ shows entrainment of frequency there is a gradient of uncoupled frequency along the organ. Although this case has been considered in previous work for the determination of limit cycle frequencies, amplitudes, and phases [7], [8], the stability of these modes has not been considered. It has been shown in Section III of this paper that the prediction of both single- and double-mode conditions is possible using the Endo and Mori method although the algebra becomes considerably more complex. The extension of this analysis to a long ladder of oscillators with a gradient of frequency is feasible but the algebra would rapidly become excessive.

The work of Endo and Mori has considered only inductive or capacitive coupling whereas in intestinal modeling it is considered that coupling is most likely to comprise both resistive and capacitive components also. The case of RLC coupled van der Pol oscillators has been considered using harmonic balance methods [9] and it has been shown that there exist regions in the RLC parameter space where only an in-phase single mode is feasible. This work can be extended to the modified fifth-power van der Pol dynamic.

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REFERENCES


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